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General Topology as Ordered Semigroup Actions. Book 1

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Todd Trimble, Andreas Blass, Robert Martin Solovay, Niels Diepeveen, and others (mentioned below) have proved some theorems which are now in this book.

Abstract. **Introduced several new axiomatic systems, that are not less general than group theory, and discovered discontinuous analysis. See [31] for an explanation why this theory is super-important.**

In this work I introduce and study in details the concepts of funcoids which generalize proximity spaces and reloids which generalize uniform spaces, and generalizations thereof. The concept of funcoid is generalized concept of proximity, the concept of reloid is cleared from superfluous details (generalized) concept of uniformity.

Also funcoids and reloids are generalizations of binary relations whose domains and ranges are filters (instead of sets). Also funcoids and reloids can be considered as a generalization of (oriented) graphs, this provides us with a common generalization of calculus and discrete mathematics.

I consider (generalized) limit of arbitrary (discontinuous) function, defined in terms of funcoids. Definition of generalized limit makes it obvious to define such things as derivative of an arbitrary function, integral of an arbitrary function, etc. It is given a definition of non-differentiable solution of a (partial) differential equation. It's raised the question how do such solutions "look like" starting a possible big future research program.

The generalized solution of one simple example differential equation is also considered.

The generalized derivatives and integrals are linear operators. For example $\int_a^b f(x)dx - \int_a^b f(x)dx = 0$ is defined and true for *every* function.

The concept of continuity is defined by an algebraic formula (instead of old messy epsilon-delta notation) for arbitrary morphisms (including funcoids and reloids) of a partially ordered category. In one formula continuity, proximity continuity, and uniform continuity are generalized.

Also I define connectedness for funcoids and reloids.

Then I consider generalizations of funcoids: pointfree funcoids and generalization of pointfree funcoids: staroids and multifuncoids. Also I define several kinds of products of funcoids and other morphisms.

I define *space* as an element of an ordered semigroup action, that is a semigroup action conforming to a partial order. Topological spaces, uniform spaces, proximity spaces, (directed) graphs, metric spaces, etc. all are spaces. It can be further generalized to ordered semicategory actions (that I call *interspaces*). I build basic general topology (continuity, limit, openness, closedness, hausdorffness, compactness, etc.) in an arbitrary space. Now general topology is an algebraic theory.

Before going to topology, this book studies properties of co-brouwerian lattices and filters.

This book was self-published under a free license by a person without scientific degree. I can't republish it in a reputable publisher, because only degree holders can receive grants.

I discovered that PhDs want to build on only on works of other PhDs. Thus by publishing it, I broke the desire of PhDs to participate in research of ordered semigroup actions.

Ordered semigroup actions may be left not researched, because no one wants to build on my research. *Oh sorry, I broke PhDs.* See [29] for more information. In other words, I did a scientific "covery": when I cover a research area from future attention, even if rediscovered.

Discovery of ordered semigroup actions is super-important by itself, but it is made finally invaluable by their missingness in scientific databases (like billion dollars for the missing cup of water in a wilderness).

Break this bond: Starts your own research based on this book. If you don't do, humanity lost ordered semigroup actions finally.

After noting that actions of ordered semigroups and discontinuous analysis are "needed" to nobody on the Earth I prayed "God, take me to heaven alone." Get this book or go to the hell.

As a scientist, you should understand that ordered semigroup actions and Discontinuous Analysis are critical for science and their non-publication draws back world economy. So, you should agree with me, that donation for publication of such discoveries may benefit you personally more than you spend, because of its multiplicative or exponential effect on the world economy. Non-donors go to the hell, because of being so much greedy that don't donate even for their own benefit.

The Apocalypse's "stamp on the forehead" (somehow related to the number 666) is when you by your forehead believe that degree (the stamp) is essential for doing science. This stamp transforms science into a stupid "beast": Your academia cannot learn even ordered semigroup actions.

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Part 1

Introductory chapters

CHAPTER 1

Introduction

The main purpose of this book is to record the current state of my research. The book is however written in such a way that it can be used as a textbook for studying my research.

For related materials, articles, research questions, and erratum consult the Web page of the author of the book:
https://math.portonvictor.org/algebraic-general-topology-and-math-synthesis/

Please consider reviewing this book at
http://www.euro-math-soc.eu/node/add/book-review

1.1. Intended audience

This book is suitable for any math student as well as for researchers.

To make this book be understandable even for first grade students, I made a chapter about basic concepts (posets, lattices, topological spaces, etc.), which an already knowledgeable person may skip reading. It is assumed that the reader knows basic set theory.

But it is also valuable for mature researchers, as it contains much original research which you could not find in any other source except of my work.

Knowledge of the basic set theory is expected from the reader.

Despite that this book presents new research, it is well structured and is suitable to be used as a textbook for a college course.

Your comments about this book are welcome to the email porton@narod.ru.

1.2. Reading Order

If you know basic order and lattice theory (including Galois connections and brouwerian lattices) and basics of category theory, you may skip reading the chapter "Common knowledge, part 1".

You are recommended to read the rest of this book by the order.

1.3. Our topic and rationale

From [45]: *Point-set topology, also called set-theoretic topology or general topology, is the study of the general abstract nature of continuity or "closeness" on spaces. Basic point-set topological notions are ones like continuity, dimension, compactness, and connectedness.*

In this work we study a new approach to point-set topology (and pointfree topology).

Traditionally general topology is studied using topological spaces (defined below in the section "Topological spaces"). I however argue that the theory of topological spaces is not the best method of studying general topology and introduce an alternative theory, the theory of *funcoids*. Despite of popularity of the theory of topological spaces it has some drawbacks and is in my opinion not the most appropriate formalism to study most of general topology. Because topological spaces are tailored for study of special sets, so called open and closed sets, studying general

topology with topological spaces is a little anti-natural and ugly. In my opinion the theory of funcoids is more elegant than the theory of topological spaces, and it is better to study funcoids than topological spaces. One of the main purposes of this work is to present an alternative General Topology based on funcoids instead of being based on topological spaces as it is customary. In order to study funcoids the prior knowledge of topological spaces is not necessary. Nevertheless in this work I will consider topological spaces and the topic of interrelation of funcoids with topological spaces.

In fact funcoids are a generalization of topological spaces, so the well known theory of topological spaces is a special case of the below presented theory of funcoids.

But probably the most important reason to study funcoids is that funcoids are a generalization of proximity spaces (see section "Proximity spaces" for the definition of proximity spaces). Before this work it was written that the theory of proximity spaces was an example of a stalled research, almost nothing interesting was discovered about this theory. It was so because the proper way to research proximity spaces is to research their generalization, funcoids. And so it was stalled until discovery of funcoids. That generalized theory of proximity spaces will bring us yet many interesting results.

In addition to *funcoids* I research *reloids*. Using below defined terminology it may be said that reloids are (basically) filters on Cartesian product of sets, and this is a special case of uniform spaces.

Afterward we study some generalizations.

Somebody might ask, why to study it? My approach relates to traditional general topology like complex numbers to real numbers theory. Be sure this will find applications.

This book has a deficiency: It does not properly relate my theory with previous research in general topology and does not consider deeper category theory properties. It is however OK for now, as I am going to do this study in later volumes (continuation of this book).

Many proofs in this book may seem too easy and thus this theory not sophisticated enough. But it is largely a result of a well structured digraph of proofs, where more difficult results are made easy by reducing them to easier lemmas and propositions.

1.4. Earlier works

Some mathematicians were researching generalizations of proximities and uniformities before me but they have failed to reach the right degree of generalization which is presented in this work allowing to represent properties of spaces with algebraic (or categorical) formulas.

Proximity structures were introduced by Smirnov in [11].

Some references to predecessors:

- In [15, 16, 25, 2, 39] generalized uniformities and proximities are studied.
- Proximities and uniformities are also studied in [22, 23, 38, 40, 41].
- [20, 21] contains recent progress in quasi-uniform spaces. [21] has a very long list of related literature.

Some works ([37]) about proximity spaces consider relationships of proximities and compact topological spaces. In this work the attempt to define or research their generalization, compactness of funcoids or reloids is not done. It seems potentially productive to attempt to borrow the definitions and procedures from the above mentioned works. I hope to do this study in a separate volume.

[10] studies mappings between proximity structures. (In this volume no attempt to research mappings between funcoids is done.) [26] researches relationships of quasi-uniform spaces and topological spaces. [1] studies how proximity structures can be treated as uniform structures and compactification regarding proximity and uniform spaces.

This book is based partially on my articles [32, 28, 30].

1.5. Kinds of continuity

A research result based on this book but not fully included in this book (and not yet published) is that the following kinds of continuity are described by the same algebraic (or rather categorical) formulas for different kinds of continuity and have common properties:

- discrete continuity (between digraphs);
- (pre)topological continuity;
- proximal continuity;
- uniform continuity;
- Cauchy continuity;
- (probably other kinds of continuity).

Thus my research justifies using the same word "continuity" for these diverse kinds of continuity.

See https://math.portonvictor.org/algebraic-general-topology-and-math-synthesis/

1.6. Responses to some accusations against style of my exposition

> The proofs are generally hard to follow and unpleasant to read as they are just a bunch of equations thrown at you, without motivation or underlying reasoning, etc.

I don't think this is essential. The proofs are not the most important thing in my book. The most essential thing are definitions. The proofs are just to fill the gaps. So I deem it not important whether proofs are motivated.

Also, note "algebraic" in the title of my book. The proofs are meant to be just sequences of formulas for as much as possible :-) It is to be thought algebraically. The meaning are the formulas themselves.

Maybe it makes sense to read my book skipping all the proofs? Some proofs contain important ideas, but most don't. The important thing are definitions.

1.7. Structure of this book

In the chapter "Common knowledge, part 1" some well known definitions and theories are considered. You may skip its reading if you already know it. That chapter contains info about:

- posets;
- lattices and complete lattices;
- Galois connections;
- co-brouwerian lattices;
- a very short intro into category theory;
- a very short introduction to group theory.

Afterward there are my little additions to poset/lattice and category theory.

Afterward there is the theory of filters and filtrators.

Then there is "Common knowledge, part 2 (topology)", which considers briefly:

- metric spaces;
- topological spaces;
- pretopological spaces;

- proximity spaces.

Despite of the name "Common knowledge" this second common knowledge chapter is recommended to be read completely even if you know topology well, because it contains some rare theorems not known to most mathematicians and hard to find in literature.

Then the most interesting thing in this book, the theory of funcoids, starts.

Afterwards there is the theory of reloids.

Then I show relationships between funcoids and reloids.

The last I research generalizations of funcoids, *pointfree funcoids*, *staroids*, and *multifuncoids* and some different kinds of products of morphisms.

1.8. Basic notation

I will denote a set definition like $\left\{\frac{x\in A}{P(x)}\right\}$ instead of customary $\{x \in A \mid P(x)\}$. I do this because otherwise formulas don't fit horizontally into the available space.

1.8.1. Grothendieck universes. We will work in ZFC with an infinite and uncountable Grothendieck universe.

A Grothendieck universe is just a set big enough to make all usual set theory inside it. For example if $\mho$ is a Grothendieck universe, and sets $X, Y \in \mho$, then also $X \cup Y \in \mho$, $X \cap Y \in \mho$, $X \times Y \in \mho$, etc.

A set which is a member of a Grothendieck universe is called a *small set* (regarding this Grothendieck universe). We can restrict our consideration to small sets in order to get rid troubles with proper classes.

Definition 1. Grothendieck universe is a set $\mho$ such that:

1°. If $x \in \mho$ and $y \in x$ then $y \in \mho$.
2°. If $x, y \in \mho$ then $\{x, y\} \in \mho$.
3°. If $x \in \mho$ then $\mathscr{P}x \in \mho$.
4°. If $\left\{\frac{x_i}{i\in I\in\mho}\right\}$ is a family of elements of $\mho$, then $\bigcup_{i\in I} x_i \in \mho$.

One can deduce from this also:

1°. If $x \in \mho$, then $\{x\} \in \mho$.
2°. If x is a subset of $y \in \mho$, then $x \in \mho$.
3°. If $x, y \in \mho$ then the ordered pair $(x, y) = \{\{x, y\}, x\} \in \mho$.
4°. If $x, y \in \mho$ then $x \cup y$ and $x \times y$ are in $\mho$.
5°. If $\left\{\frac{x_i}{i\in I\in\mho}\right\}$ is a family of elements of $\mho$, then the product $\prod_{i\in I} x_i \in \mho$.
6°. If $x \in \mho$, then the cardinality of x is strictly less than the cardinality of $\mho$.

1.8.2. Misc. In this book quantifiers bind tightly. That is $\forall x \in A : P(x) \wedge Q$ and $\forall x \in A : P(x) \Rightarrow Q$ should be read $(\forall x \in A : P(x)) \wedge Q$ and $(\forall x \in A : P(x)) \Rightarrow Q$ not $\forall x \in A : (P(x) \wedge Q)$ and $\forall x \in A : (P(x) \Rightarrow Q)$.

The set of functions from a set A to a set B is denoted as B^A.

I will often skip parentheses and write fx instead of $f(x)$ to denote the result of a function f acting on the argument x.

I will denote $\langle f\rangle^* X = \left\{\frac{\beta\in\operatorname{im} f}{\exists\alpha\in X:\alpha f\beta}\right\}$ (in other words $\langle f\rangle^* X$ is the image of a set X under a function or binary relation f) and $X\ [f]^*\ Y \Leftrightarrow \exists x \in X, y \in Y : x\ f\ y$ for sets X, Y and a binary relation f. (Note that functions are a special case of binary relations.)

By just $\langle f\rangle^*$ and $[f]^*$ I will denote the corresponding function and relation on small sets.

Obvious 2. For a function f we have $\langle f\rangle^* X = \left\{\frac{f(x)}{x\in X}\right\}$.

DEFINITION 3. $\left\langle f^{-1} \right\rangle^* X$ is called the *preimage* of a set X by a function (or, more generally, a binary relation) f.

OBVIOUS 4. $\{\alpha\} [f]^* \{\beta\} \Leftrightarrow \alpha \; f \; \beta$ for every α and β.

$\lambda x \in D : f(x) = \left\{ \frac{(x, f(x))}{x \in D} \right\}$ for a set D and and a form f depending on the variable x. In other words, $\lambda x \in D : f(x)$ is the function which maps elements x of a set D into $f(x)$.

I will denote source and destination of a morphism f of any category (See chapter 2 for a definition of a category.) as $\operatorname{Src} f$ and $\operatorname{Dst} f$ correspondingly. Note that below defined domain and image of a funcoid are not the same as its source and destination.

I will denote $\operatorname{GR}(A, B, f) = f$ for any morphism (A, B, f) of either **Set** or **Rel**. (See definitions of **Set** and **Rel** below.)

1.9. Implicit arguments

Some notation such that $\bot^{\mathfrak{A}}$, $\top^{\mathfrak{A}}$, $\sqcup^{\mathfrak{A}}$, $\sqcap^{\mathfrak{A}}$ have indexes (in these examples $\mathfrak{A}$).

We will omit these indexes when they can be restored from the context. For example, having a function $f : \mathfrak{A} \to \mathfrak{B}$ where $\mathfrak{A}$, $\mathfrak{B}$ are posets with least elements, we will concisely denote $f\bot = \bot$ for $f\bot^{\mathfrak{A}} = \bot^{\mathfrak{B}}$. (See below for definitions of these operations.)

NOTE 5. In the above formula $f\bot = \bot$ we have the first $\bot$ and the second $\bot$ denoting different objects.

We will assume (skipping this in actual proofs) that all omitted indexes can be restored from context. (Note that so called dependent type theory computer proof assistants do this like we implicitly.)

1.10. Unusual notation

In the chapter "Common knowledge, part 1" (which you may skip reading if you are already knowledgeable) some non-standard notation is defined. I summarize here this notation for the case if you choose to skip reading that chapter:

Partial order is denoted as $\sqsubseteq$.

Meets and joins are denoted as $\sqcap$, $\sqcup$, $\bigsqcap$, $\bigsqcup$.

I call element b *substractive* from an element a (of a distributive lattice $\mathfrak{A}$) when the difference $a \setminus b$ exists. I call b *complementive* to a when there exists $c \in \mathfrak{A}$ such that $b \sqcap c = \bot$ and $b \sqcup c = a$. We will prove that b is complementive to a iff b is substractive from a and $b \sqsubseteq a$.

DEFINITION 6. Call a and b of a poset $\mathfrak{A}$ *intersecting*, denoted $a \not\asymp b$, when there exists a non-least element c such that $c \sqsubseteq a \wedge c \sqsubseteq b$.

DEFINITION 7. $a \asymp b \overset{\text{def}}{=} \neg(a \not\asymp b)$.

DEFINITION 8. I call elements a and b of a poset $\mathfrak{A}$ *joining* and denote $a \equiv b$ when there are no non-greatest element c such that $c \sqsupseteq a \wedge c \sqsupseteq b$.

DEFINITION 9. $a \not\equiv b \overset{\text{def}}{=} \neg(a \equiv b)$.

OBVIOUS 10. $a \not\asymp b$ iff $a \sqcap b$ is non-least, for every elements a, b of a meet-semilattice.

OBVIOUS 11. $a \equiv b$ iff $a \sqcup b$ is the greatest element, for every elements a, b of a join-semilattice.

I extend the definitions of pseudocomplement and dual pseudocomplement to arbitrary posets (not just lattices as it is customary):

Definition 12. Let $\mathfrak{A}$ be a poset. *Pseudocomplement* of a is

$$\max\left\{\frac{c \in \mathfrak{A}}{c \asymp a}\right\}.$$

If z is the pseudocomplement of a we will denote $z = a^*$.

Definition 13. Let $\mathfrak{A}$ be a poset. *Dual pseudocomplement* of a is

$$\min\left\{\frac{c \in \mathfrak{A}}{c \equiv a}\right\}.$$

If z is the dual pseudocomplement of a we will denote $z = a^+$.

CHAPTER 2

Common knowledge, part 1

In this chapter we will consider some well known mathematical theories. If you already know them you may skip reading this chapter (or its parts).

2.1. Order theory

2.1.1. Posets.

Definition 14. The *identity relation* on a set A is $\mathrm{id}_A = \left\{ \frac{(a,a)}{a \in A} \right\}$.

Definition 15. A *preorder* on a set A is a binary relation $\sqsubseteq$ on A which is:

- *reflexive* on A that is $(\sqsubseteq) \supseteq \mathrm{id}_A$ or what is the same $\forall x \in A : x \sqsubseteq x$;
- *transitive* that is $(\sqsubseteq) \circ (\sqsubseteq) \subseteq (\sqsubseteq)$ or what is the same

$$\forall x, y, z : (x \sqsubseteq y \wedge y \sqsubseteq z \Rightarrow x \sqsubseteq z).$$

Definition 16. A *partial order* on a set A is a preorder on A which is *anti-symmetric* that is $(\sqsubseteq) \cap (\sqsubseteq) \subseteq \mathrm{id}_A$ or what is the same

$$\forall x, y \in A : (x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y).$$

The reverse relation is denoted $\sqsupseteq$.

Definition 17. a is a subelement of b (or what is the same a is *contained* in b or b *contains* a) iff $a \sqsubseteq b$.

Obvious 18. The reverse of a partial order is also a partial order.

Definition 19. A set A together with a partial order on it is called a *partially ordered set* (*poset* for short).

An example of a poset is the set $\mathbb{R}$ of real numbers with $\sqsubseteq = \leq$.

Another example is the set $\mathscr{P}A$ of all subsets of an arbitrary fixed set A with $\sqsubseteq = \subseteq$. Note that this poset is (in general) not linear (see definition of *linear* poset below.)

Definition 20. Strict partial order $\sqsubset$ corresponding to the partial order $\sqsubseteq$ on a set A is defined by the formula $(\sqsubset) = (\sqsubseteq) \setminus \mathrm{id}_A$. In other words,

$$a \sqsubset b \Leftrightarrow a \sqsubseteq b \wedge a \neq b.$$

An example of strict partial order is $<$ on the set $\mathbb{R}$ of real numbers.

Definition 21. A partial order on a set A *restricted* to a set $B \subseteq A$ is $(\sqsubseteq) \cap (B \times B)$.

Obvious 22. A partial order on a set A restricted to a set $B \subseteq A$ is a partial order on B.

Definition 23.

- The *least* element $\perp$ of a poset $\mathfrak{A}$ is defined by the formula $\forall a \in \mathfrak{A} : \perp \sqsubseteq a$.
- The *greatest* element $\top$ of a poset $\mathfrak{A}$ is defined by the formula $\forall a \in \mathfrak{A} : \top \sqsupseteq a$.

PROPOSITION 24. There exist no more than one least element and no more than one greatest element (for a given poset).

PROOF. By antisymmetry. □

DEFINITION 25. The *dual* order for $\sqsubseteq$ is $\sqsupseteq$.

OBVIOUS 26. Dual of a partial order is a partial order.

DEFINITION 27. The *dual* poset for a poset $(A, \sqsubseteq)$ is the poset $(A, \sqsupseteq)$.

I will denote dual of a poset $\mathfrak{A}$ as $(\operatorname{dual} \mathfrak{A})$ and dual of an element $a \in \mathfrak{A}$ (that is the same element in the dual poset) as $(\operatorname{dual} a)$.

Below we will sometimes use *duality* that is replacement of the partial order and all related operations and relations with their duals. In other words, it is enough to prove a theorem for an order $\sqsubseteq$ and the similar theorem for $\sqsupseteq$ follows by duality.

DEFINITION 28. A subset P of a poset $\mathfrak{A}$ is called *bounded above* if there exists $t \in \mathfrak{A}$ such that $\forall x \in P : t \sqsupseteq x$. *Bounded below* is defined dually.

2.1.1.1. *Intersecting and joining elements.* Let $\mathfrak{A}$ be a poset.

DEFINITION 29. Call elements a and b of $\mathfrak{A}$ *intersecting*, denoted $a \not\asymp b$, when there exists a non-least element c such that $c \sqsubseteq a \wedge c \sqsubseteq b$.

DEFINITION 30. $a \asymp b \stackrel{\text{def}}{=} \neg(a \not\asymp b)$.

OBVIOUS 31. $a_0 \not\asymp b_0 \wedge a_1 \sqsupseteq a_0 \wedge b_1 \sqsupseteq b_0 \Rightarrow a_1 \not\asymp b_1$.

DEFINITION 32. I call elements a and b of $\mathfrak{A}$ *joining* and denote $a \equiv b$ when there is no a non-greatest element c such that $c \sqsupseteq a \wedge c \sqsupseteq b$.

DEFINITION 33. $a \not\equiv b \stackrel{\text{def}}{=} \neg(a \equiv b)$.

OBVIOUS 34. Intersecting is the dual of non-joining.

OBVIOUS 35. $a_0 \equiv b_0 \wedge a_1 \sqsupseteq a_0 \wedge b_1 \sqsupseteq b_0 \Rightarrow a_1 \equiv b_1$.

2.1.2. Linear order.

DEFINITION 36. A poset $\mathfrak{A}$ is called *linearly ordered set* (or what is the same, *totally ordered set*) if $a \sqsupseteq b \vee b \sqsupseteq a$ for every $a, b \in \mathfrak{A}$.

EXAMPLE 37. The set of real numbers with the customary order is a linearly ordered set.

DEFINITION 38. A set $X \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a poset is called *chain* if $\mathfrak{A}$ restricted to X is a total order.

2.1.3. Meets and joins. Let $\mathfrak{A}$ be a poset.

DEFINITION 39. Given a set $X \in \mathscr{P}\mathfrak{A}$ the *least element* (also called *minimum* and denoted $\min X$) of X is such $a \in X$ that $\forall x \in X : a \sqsubseteq x$.

Least element does not necessarily exists. But if it exists:

PROPOSITION 40. For a given $X \in \mathscr{P}\mathfrak{A}$ there exist no more than one least element.

PROOF. It follows from anti-symmetry. □

Greatest element is the dual of least element:

DEFINITION 41. Given a set $X \in \mathscr{P}\mathfrak{A}$ the *greatest element* (also called *maximum* and denoted $\max X$) of X is such $a \in X$ that $\forall x \in X : a \sqsupseteq x$.

Remark 42. Least and greatest elements of a set X is a trivial generalization of the above defined least and greatest element for the entire poset.

Definition 43.

- A *minimal* element of a set $X \in \mathscr{P}\mathfrak{A}$ is such $a \in \mathfrak{A}$ that $\nexists x \in X : a \sqsupset x$.
- A *maximal* element of a set $X \in \mathscr{P}\mathfrak{A}$ is such $a \in \mathfrak{A}$ that $\nexists x \in X : a \sqsubset x$.

Remark 44. Minimal element is not the same as minimum, and maximal element is not the same as maximum.

Obvious 45.

1°. The least element (if it exists) is a minimal element.
2°. The greatest element (if it exists) is a maximal element.

Exercise 46. Show that there may be more than one minimal and more than one maximal element for some poset.

Definition 47. *Upper bounds* of a set X is the set $\left\{ \frac{y \in \mathfrak{A}}{\forall x \in X : y \sqsupseteq x} \right\}$.

The dual notion:

Definition 48. *Lower bounds* of a set X is the set $\left\{ \frac{y \in \mathfrak{A}}{\forall x \in X : y \sqsubseteq x} \right\}$.

Definition 49. *Join* $\bigsqcup X$ (also called *supremum* and denoted "$\sup X$") of a set X is the least element of its upper bounds (if it exists).

Definition 50. *Meet* $\bigsqcap X$ (also called *infimum* and denoted "$\inf X$") of a set X is the greatest element of its lower bounds (if it exists).

We will also denote $\bigsqcup_{i \in X} f(i) = \bigsqcup \left\{ \frac{f(i)}{x \in X} \right\}$ and $\bigsqcap_{i \in X} f(i) = \bigsqcap \left\{ \frac{f(i)}{x \in X} \right\}$.

We will write $b = \bigsqcup X$ when $b \in \mathfrak{A}$ is the join of X or say that $\bigsqcup X$ does not exist if there are no such $b \in \mathfrak{A}$. (And dually for meets.)

Exercise 51. Provide an example of $\bigsqcup X \notin X$ for some set X on some poset.

Proposition 52.

1°. If b is the greatest element of X then $\bigsqcup X = b$.
2°. If b is the least element of X then $\bigsqcap X = b$.

Proof. We will prove only the first as the second is dual.

Let b be the greatest element of X. Then upper bounds of X are $\left\{ \frac{y \in \mathfrak{A}}{y \sqsupseteq b} \right\}$. Obviously b is the least element of this set, that is the join. □

Definition 53. *Binary joins and meets* are defined by the formulas

$$x \sqcup y = \bigsqcup \{x, y\} \quad \text{and} \quad x \sqcup y = \bigsqcap \{x, y\}.$$

Obvious 54. $\sqcup$ and $\sqcap$ are symmetric operations (whenever these are defined for given x and y).

Theorem 55.

1°. If $\bigsqcup X$ exists then $y \sqsupseteq \bigsqcup X \Leftrightarrow \forall x \in X : y \sqsupseteq x$.
2°. If $\bigsqcap X$ exists then $y \sqsubseteq \bigsqcap X \Leftrightarrow \forall x \in X : y \sqsubseteq x$.

Proof. I will prove only the first as the second follows by duality.

$y \sqsupseteq \bigsqcup X \Leftrightarrow y$ is an upper bound for $X \Leftrightarrow \forall x \in X : y \sqsupseteq x$. □

Corollary 56.

1°. If $a \sqcup b$ exists then $y \sqsupseteq a \sqcup b \Leftrightarrow y \sqsupseteq a \wedge y \sqsupseteq b$.
2°. If $a \sqcap b$ exists then $y \sqsubseteq a \sqcap b \Leftrightarrow y \sqsubseteq a \wedge y \sqsubseteq b$.

I will denote meets and joins for a specific poset $\mathfrak{A}$ as $\bigsqcap^{\mathfrak{A}}$, $\bigsqcup^{\mathfrak{A}}$, $\sqcap^{\mathfrak{A}}$, $\sqcup^{\mathfrak{A}}$.

2.1.4. Semilattices.

Definition 57.

1°. A *join-semilattice* is a poset $\mathfrak{A}$ such that $a \sqcup b$ is defined for every $a, b \in \mathfrak{A}$.
2°. A *meet-semilattice* is a poset $\mathfrak{A}$ such that $a \sqcap b$ is defined for every $a, b \in \mathfrak{A}$.

Theorem 58.

1°. The operation $\sqcup$ is associative for any join-semilattice.
2°. The operation $\sqcap$ is associative for any meet-semilattice.

Proof. I will prove only the first as the second follows by duality.
We need to prove $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$ for every $a, b, c \in \mathfrak{A}$.
Taking into account the definition of join, it is enough to prove that

$$x \sqsupseteq (a \sqcup b) \sqcup c \Leftrightarrow x \sqsupseteq a \sqcup (b \sqcup c)$$

for every $x \in \mathfrak{A}$. Really, this follows from the chain of equivalences:

$$\begin{aligned} x \sqsupseteq (a \sqcup b) \sqcup c &\Leftrightarrow \\ x \sqsupseteq a \sqcup b \wedge x \sqsupseteq c &\Leftrightarrow \\ x \sqsupseteq a \wedge x \sqsupseteq b \wedge x \sqsupseteq c &\Leftrightarrow \\ x \sqsupseteq a \wedge x \sqsupseteq b \sqcup c &\Leftrightarrow \\ x \sqsupseteq a \sqcup (b \sqcup c). & \end{aligned}$$

□

Obvious 59. $a \not\asymp b$ iff $a \sqcap b$ is non-least, for every elements a, b of a meet-semilattice.

Obvious 60. $a \equiv b$ iff $a \sqcup b$ is the greatest element, for every elements a, b of a join-semilattice.

2.1.5. Lattices and complete lattices.

Definition 61. A *bounded* poset is a poset having both least and greatest elements.

Definition 62. *Lattice* is a poset which is both join-semilattice and meet-semilattice.

Definition 63. A *complete lattice* is a poset $\mathfrak{A}$ such that for every $X \in \mathscr{P}\mathfrak{A}$ both $\bigsqcup X$ and $\bigsqcap X$ exist.

Obvious 64. Every complete lattice is a lattice.

Proposition 65. Every complete lattice is a bounded poset.

Proof. $\bigsqcup \emptyset$ is the least and $\bigsqcap \emptyset$ is the greatest element. □

Theorem 66. Let $\mathfrak{A}$ be a poset.

1°. If $\bigsqcup X$ is defined for every $X \in \mathscr{P}\mathfrak{A}$, then $\mathfrak{A}$ is a complete lattice.
2°. If $\bigsqcap X$ is defined for every $X \in \mathscr{P}\mathfrak{A}$, then $\mathfrak{A}$ is a complete lattice.

Proof. See [27] or any lattice theory reference. □

Obvious 67. If $X \subseteq Y$ for some $X, Y \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a complete lattice, then

1°. $\bigsqcup X \sqsubseteq \bigsqcup Y$;
2°. $\bigsqcap X \sqsupseteq \bigsqcap Y$.

Proposition 68. If $S \in \mathscr{P}\mathscr{P}\mathfrak{A}$ then for every complete lattice $\mathfrak{A}$

1°. $\bigsqcup\bigcup S = \bigsqcup_{X\in S}\bigsqcup X$;
2°. $\bigsqcap\bigcup S = \bigsqcap_{X\in S}\bigsqcap X$.

Proof. We will prove only the first as the second is dual.
By definition of joins, it is enough to prove $y \sqsupseteq \bigsqcup\bigcup S \Leftrightarrow y \sqsupseteq \bigsqcup_{X\in S}\bigsqcup X$.
Really,

$$\begin{aligned} y \sqsupseteq \bigsqcup\bigcup S \Leftrightarrow \\ \forall x \in \bigcup S : y \sqsupseteq x \Leftrightarrow \\ \forall X \in S \forall x \in X : y \sqsupseteq x \Leftrightarrow \\ \forall X \in S : y \sqsupseteq \bigsqcup X \Leftrightarrow \\ y \sqsupseteq \bigsqcup_{X\in S}\bigsqcup X. \end{aligned}$$

□

Definition 69. A *sublattice* of a lattice is it subset closed regarding $\sqcup$ and $\sqcap$.

Obvious 70. Sublattice with induced order is also a lattice.

2.1.6. Distributivity of lattices.

Definition 71. A *distributive* lattice is such lattice $\mathfrak{A}$ that for every $x, y, z \in \mathfrak{A}$
1°. $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$;
2°. $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$.

Theorem 72. For a lattice to be distributive it is enough just one of the conditions:
1°. $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$;
2°. $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$.

Proof.

$$\begin{aligned} (x \sqcup y) \sqcap (x \sqcup z) = \\ ((x \sqcup y) \sqcap x) \sqcup ((x \sqcup y) \sqcap z) = \\ x \sqcup ((x \sqcap z) \sqcup (y \sqcap z)) = \\ (x \sqcup (x \sqcap z)) \sqcup (y \sqcap z) = \\ x \sqcup (y \sqcap z) \end{aligned}$$

(applied $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$ twice). □

2.1.7. Difference and complement.

Definition 73. Let $\mathfrak{A}$ be a distributive lattice with least element $\perp$. The *difference* (denoted $a \setminus b$) of elements a and b is such $c \in \mathfrak{A}$ that $b \sqcap c = \perp$ and $a \sqcup b = b \sqcup c$. I will call b *substractive* from a when $a \setminus b$ exists.

Theorem 74. If $\mathfrak{A}$ is a distributive lattice with least element $\perp$, there exists no more than one difference of elements a, b.

Proof. Let c and d be both differences $a \setminus b$. Then $b \sqcap c = b \sqcap d = \perp$ and $a \sqcup b = b \sqcup c = b \sqcup d$. So

$$c = c \sqcap (b \sqcup c) = c \sqcap (b \sqcup d) = (c \sqcap b) \sqcup (c \sqcap d) = \perp \sqcup (c \sqcap d) = c \sqcap d.$$

Similarly $d = d \sqcap c$. Consequently $c = c \sqcap d = d \sqcap c = d$. □

Definition 75. I will call b *complementive* to a iff there exists $c \in \mathfrak{A}$ such that $b \sqcap c = \bot$ and $b \sqcup c = a$.

Proposition 76. b is complementive to a iff b is substractive from a and $b \sqsubseteq a$.

Proof.

$\Leftarrow$. Obvious.
$\Rightarrow$. We deduce $b \sqsubseteq a$ from $b \sqcup c = a$. Thus $a \sqcup b = a = b \sqcup c$.

□

Proposition 77. If b is complementive to a then $(a \setminus b) \sqcup b = a$.

Proof. Because $b \sqsubseteq a$ by the previous proposition. □

Definition 78. Let $\mathfrak{A}$ be a bounded distributive lattice. The *complement* (denoted $\bar{a}$) of an element $a \in \mathfrak{A}$ is such $b \in \mathfrak{A}$ that $a \sqcap b = \bot$ and $a \sqcup b = \top$.

Proposition 79. If $\mathfrak{A}$ is a bounded distributive lattice then $\bar{a} = \top \setminus a$.

Proof. $b = \bar{a} \Leftrightarrow b \sqcap a = \bot \wedge b \sqcup a = \top \Leftrightarrow b \sqcap a = \bot \wedge \top \sqcup a = a \sqcup b \Leftrightarrow b = \top \setminus a$. □

Corollary 80. If $\mathfrak{A}$ is a bounded distributive lattice then exists no more than one complement of an element $a \in \mathfrak{A}$.

Definition 81. An element of bounded distributive lattice is called *complemented* when its complement exists.

Definition 82. A distributive lattice is a *complemented lattice* iff every its element is complemented.

Proposition 83. For a distributive lattice $(a \setminus b) \setminus c = a \setminus (b \sqcup c)$ if $a \setminus b$ and $(a \setminus b) \setminus c$ are defined.

Proof. $((a \setminus b) \setminus c) \sqcap c = \bot$; $((a \setminus b) \setminus c) \sqcup c = (a \setminus b) \sqcup c$; $(a \setminus b) \sqcap b = \bot$; $(a \setminus b) \sqcup b = a \sqcup b$.

We need to prove $((a \setminus b) \setminus c) \sqcap (b \sqcup c) = \bot$ and $((a \setminus b) \setminus c) \sqcup (b \sqcup c) = a \sqcup (b \sqcup c)$.

In fact,

$$\begin{aligned}
((a \setminus b) \setminus c) \sqcap (b \sqcup c) &= \\
(((a \setminus b) \setminus c) \sqcap b) \sqcup (((a \setminus b) \setminus c) \sqcap c) &= \\
(((a \setminus b) \setminus c) \sqcap b) \sqcup \bot &= \\
((a \setminus b) \setminus c) \sqcap b &\sqsubseteq \\
(a \setminus b) \sqcap b &= \bot,
\end{aligned}$$

so $((a \setminus b) \setminus c) \sqcap (b \sqcup c) = \bot$;

$$\begin{aligned}
((a \setminus b) \setminus c) \sqcup (b \sqcup c) &= \\
(((a \setminus b) \setminus c) \sqcup c) \sqcup b &= \\
(a \setminus b) \sqcup c \sqcup b &= \\
((a \setminus b) \sqcup b) \sqcup c &= \\
a \sqcup b \sqcup c.
\end{aligned}$$

□

2.1.8. Boolean lattices.

Definition 84. A *boolean lattice* is a complemented distributive lattice.

The most important example of a boolean lattice is $\mathscr{P}A$ where A is a set, ordered by set inclusion.

Theorem 85. (De Morgan's laws) For every elements a, b of a boolean lattice
1°. $\overline{a \sqcup b} = \bar{a} \sqcap \bar{b}$;
2°. $\overline{a \sqcap b} = \bar{a} \sqcup b$.

Proof. We will prove only the first as the second is dual.
It is enough to prove that $a \sqcup b$ is a complement of $\bar{a} \sqcap \bar{b}$. Really:

$$(a \sqcup b) \sqcap (\bar{a} \sqcap \bar{b}) \sqsubseteq a \sqcap (\bar{a} \sqcap \bar{b}) = (a \sqcap \bar{a}) \sqcap \bar{b} = \bot \sqcap \bar{b} = \bot;$$

$$(a \sqcup b) \sqcup (\bar{a} \sqcap \bar{b}) = ((a \sqcup b) \sqcup \bar{a}) \sqcap ((a \sqcup b) \sqcup \bar{b}) \sqsupseteq (a \sqcup \bar{a}) \sqcap (b \sqcup \bar{b}) = \top \sqcap \top = \top.$$

Thus $(a \sqcup b) \sqcap (\bar{a} \sqcap \bar{b}) = \bot$ and $(a \sqcup b) \sqcup (\bar{a} \sqcap \bar{b}) = \top$. □

Definition 86. A complete lattice $\mathfrak{A}$ is *join infinite distributive* when $x \sqcap \bigsqcup S = \bigsqcup \langle x \sqcap \rangle^* S$; a complete lattice $\mathfrak{A}$ is *meet infinite distributive* when $x \sqcup \bigsqcap S = \bigsqcap \langle x \sqcup \rangle^* S$ for all $x \in \mathfrak{A}$ and $S \in \mathscr{P}\mathfrak{A}$.

Definition 87. *Infinite distributive complete lattice* is a complete lattice which is both join infinite distributive and meet infinite distributive.

Theorem 88. For every boolean lattice $\mathfrak{A}$, $x \in \mathfrak{A}$ and $S \in \mathscr{P}\mathfrak{A}$ we have:
1°. $\bigsqcup \langle x \sqcap \rangle^* S$ is defined and $x \sqcap \bigsqcup S = \bigsqcup \langle x \sqcap \rangle^* S$ whenever $\bigsqcup S$ is defined.
2°. $\bigsqcap \langle x \sqcup \rangle^* S$ is defined and $x \sqcup \bigsqcap S = \bigsqcap \langle x \sqcup \rangle^* S$ whenever $\bigsqcap S$ is defined.

Proof. We will prove only the first, as the other is dual.
We need to prove that $x \sqcap \bigsqcup S$ is the least upper bound of $\langle x \sqcap \rangle^* S$.
That $x \sqcap \bigsqcup S$ is an upper bound of $\langle x \sqcap \rangle^* S$ is obvious.
Now let u be any upper bound of $\langle x \sqcap \rangle^* S$, that is $x \sqcap y \sqsubseteq u$ for all $y \in S$. Then

$$y = y \sqcap (x \sqcup \bar{x}) = (y \sqcap x) \sqcup (y \sqcap \bar{x}) \sqsubseteq u \sqcup \bar{x},$$

and so $\bigsqcup S \sqsubseteq u \sqcup \bar{x}$. Thus

$$x \sqcap \bigsqcup S \sqsubseteq x \sqcap (u \sqcup \bar{x}) = (x \sqcap u) \sqcup (x \sqcap \bar{x}) = (x \sqcap u) \sqcup \bot = x \sqcap u \sqsubseteq u,$$

that is $x \sqcap \bigsqcup S$ is the least upper bound of $\langle x \sqcap \rangle^* S$. □

Corollary 89. Every complete boolean lattice is both join infinite distributive and meet infinite distributive.

Theorem 90. (infinite De Morgan's laws) For every subset S of a complete boolean lattice
1°. $\overline{\bigsqcup S} = \bigsqcap_{x \in S} \bar{x}$;
2°. $\overline{\bigsqcap S} = \bigsqcup_{x \in S} \bar{x}$.

Proof. It's enough to prove that $\bigsqcup S$ is a complement of $\bigsqcap_{x \in S} \bar{x}$ (the second follows from duality). Really, using the previous theorem:

$$\bigsqcup S \sqcup \bigsqcap_{x \in S} \bar{x} = \bigsqcap_{x \in S} \Big\langle \bigsqcup S \sqcup \Big\rangle^* \bar{x} = \bigsqcap \left\{ \frac{\bigsqcup S \sqcup \bar{x}}{x \in S} \right\} \sqsupseteq \bigsqcap \left\{ \frac{x \sqcup \bar{x}}{x \in S} \right\} = \top;$$

$$\bigsqcup S \sqcap \bigsqcap_{x \in S} \bar{x} = \bigsqcup_{y \in S} \left\langle \bigsqcap_{x \in S} \bar{x} \sqcap \right\rangle^* y = \bigsqcup \left\{ \frac{\bigsqcap_{x \in S} \bar{x} \sqcap y}{y \in S} \right\} \sqsubseteq \bigsqcup \left\{ \frac{\bar{y} \sqcap y}{y \in S} \right\} = \bot.$$

So $\bigsqcup S \sqcup \bigsqcap_{x \in S} \bar{x} = \top$ and $\bigsqcup S \sqcap \bigsqcap_{x \in S} \bar{x} = \bot$. □

2.1.9. Center of a lattice.

Definition 91. The *center* $Z(\mathfrak{A})$ of a bounded distributive lattice $\mathfrak{A}$ is the set of its complemented elements.

Remark 92. For a definition of center of non-distributive lattices see [5].

Remark 93. In [24] the word center and the notation $Z(\mathfrak{A})$ are used in a different sense.

Definition 94. A sublattice K of a complete lattice L is a *closed sublattice* of L if K contains the meet and the join of any its nonempty subset.

Theorem 95. Center of an infinitely distributive lattice is its closed sublattice.

Proof. See [17]. □

Remark 96. See [18] for a more strong result.

Theorem 97. The center of a bounded distributive lattice constitutes its sublattice.

Proof. Let $\mathfrak{A}$ be a bounded distributive lattice and $Z(\mathfrak{A})$ be its center. Let $a, b \in Z(\mathfrak{A})$. Consequently $\bar{a}, \bar{b} \in Z(\mathfrak{A})$. Then $\bar{a} \sqcup \bar{b}$ is the complement of $a \sqcap b$ because

$$(a \sqcap b) \sqcap (\bar{a} \sqcup \bar{b}) = (a \sqcap b \sqcap \bar{a}) \sqcup (a \sqcap b \sqcap \bar{b}) = \bot \sqcup \bot = \bot \quad \text{and}$$
$$(a \sqcap b) \sqcup (\bar{a} \sqcup \bar{b}) = (a \sqcup \bar{a} \sqcup \bar{b}) \sqcap (b \sqcup \bar{a} \sqcup \bar{b}) = \top \sqcap \top = \top.$$

So $a \sqcap b$ is complemented. Similarly $a \sqcup b$ is complemented. □

Theorem 98. The center of a bounded distributive lattice constitutes a boolean lattice.

Proof. Because it is a distributive complemented lattice. □

2.1.10. Atoms of posets.

Definition 99. An atom of a poset is an element a such that (for every its element x) $x \sqsubset a$ if and only if x is the least element.

Remark 100. This definition is valid even for posets without least element.

Proposition 101. Element a is an atom iff both:

1°. $x \sqsubset a$ implies x is the least element;
2°. a is non-least.

Proof.

⇒. Let a be an atom. 1° is obvious. If a is least then $a \sqsubset a$ what is impossible, so 2°.

⇐. Let 1° and 2° hold. We need to prove only that x is least implies that $x \sqsubset a$ but this follows from a being non-least.

□

Example 102. Atoms of the boolean algebra $\mathscr{P}A$ (ordered by set inclusion) are one-element sets.

I will denote $\operatorname{atoms}^{\mathfrak{A}} a$ or just $(\operatorname{atoms} a)$ the set of atoms contained in an element a of a poset $\mathfrak{A}$. I will denote $\operatorname{atoms}^{\mathfrak{A}}$ the set of all atoms of a poset $\mathfrak{A}$.

Definition 103. A poset $\mathfrak{A}$ is called *atomic* iff $\operatorname{atoms} a \neq \emptyset$ for every non-least element a of the poset $\mathfrak{A}$.

DEFINITION 104. *Atomistic poset* is such a poset that $a = \bigsqcup \operatorname{atoms} a$ for every element a of this poset.

OBVIOUS 105. Every atomistic poset is atomic.

PROPOSITION 106. Let $\mathfrak{A}$ be a poset. If a is an atom of $\mathfrak{A}$ and $B \in \mathfrak{A}$ then

$$a \in \operatorname{atoms} B \Leftrightarrow a \sqsubseteq B \Leftrightarrow a \not\asymp B.$$

PROOF.

$a \in \operatorname{atoms} B \Leftrightarrow a \sqsubseteq B$. Obvious.

$a \sqsubseteq B \Rightarrow a \not\asymp B$. $a \sqsubseteq B \Rightarrow a \sqsubseteq a \wedge a \sqsubseteq B$, thus $a \not\asymp B$ because a is not least.

$a \sqsubseteq B \Leftarrow a \not\asymp B$. $a \not\asymp B$ implies existence of non-least element x such that $x \sqsubseteq B$ and $x \sqsubseteq a$. Because a is an atom, we have $x = a$. So $a \sqsubseteq B$.

□

THEOREM 107. A poset is atomistic iff every its element can be represented as join of atoms.

PROOF.

$\Rightarrow$. Obvious.

$\Leftarrow$. Let $a = \bigsqcup S$ where S is a set of atoms. We will prove that a is the least upper bound of $\operatorname{atoms} a$.

That a is an upper bound of $\operatorname{atoms} a$ is obvious. Let x is an upper bound of $\operatorname{atoms} a$. Then $x \sqsupseteq \bigsqcup S$ because $S \subseteq \operatorname{atoms} a$. Thus $x \sqsupseteq a$.

□

THEOREM 108. $\operatorname{atoms} \bigsqcap S = \bigcap \langle \operatorname{atoms} \rangle^* S$ whenever $\bigsqcap S$ is defined for every $S \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a poset.

PROOF. For any atom

$$\begin{aligned} c \in \operatorname{atoms} \bigsqcap S &\Leftrightarrow \\ c \sqsubseteq \bigsqcap S &\Leftrightarrow \\ \forall a \in S : c \sqsubseteq a &\Leftrightarrow \\ \forall a \in S : c \in \operatorname{atoms} a &\Leftrightarrow \\ c \in \bigcap \langle \operatorname{atoms} \rangle^* S. & \end{aligned}$$

□

COROLLARY 109. $\operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$ for an arbitrary meet-semilattice.

THEOREM 110. A complete boolean lattice is atomic iff it is atomistic.

PROOF.

$\Leftarrow$. Obvious.

$\Rightarrow$. Let $\mathfrak{A}$ be an atomic boolean lattice. Let $a \in \mathfrak{A}$. Suppose $b = \bigsqcup \operatorname{atoms} a \sqsubset a$. If $x \in \operatorname{atoms}(a \setminus b)$ then $x \sqsubseteq a \setminus b$ and so $x \sqsubseteq a$ and hence $x \sqsubseteq b$. But we have $x = x \sqcap b \sqsubseteq (a \setminus b) \sqcap b = \bot$ what contradicts to our supposition.

□

2.1.11. Kuratowski's lemma.

THEOREM 111. (KURATOWSKI's lemma) Any chain in a poset is contained in a maximal chain (if we order chains by inclusion).

I will skip the proof of KURATOWSKI's lemma as this proof can be found in any set theory or order theory reference.

2.1.12. Homomorphisms of posets and lattices.

Definition 112. A *monotone* function (also called *order homomorphism*) from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ is such a function f that $x \sqsubseteq y \Rightarrow fx \sqsubseteq fy$ for every $x, y \in \mathfrak{A}$.

Definition 113. A *antitone* function (also called *antitone order homomorphism*) from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ is such a function f that $x \sqsubseteq y \Rightarrow fx \sqsupseteq fy$ for every $x, y \in \mathfrak{A}$.

Definition 114. *Order embedding* is a function f from poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ such that $x \sqsubseteq y \Leftrightarrow fx \sqsubseteq fy$ for every $x, y \in \mathfrak{A}$.

Proposition 115. Every order embedding is injective.

Proof. $fx = fy$ implies $x \sqsubseteq y$ and $y \sqsubseteq x$. □

Obvious 116. Every order embedding is an order homomorphism.

Definition 117. *Antitone order embedding* is a function f from poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ such that $x \sqsubseteq y \Leftrightarrow fx \sqsupseteq fy$ for every $x, y \in \mathfrak{A}$.

Obvious 118. Antitone order embedding is an order embedding between a poset and a dual of (another) poset.

Definition 119. *Order isomorphism* is a surjective order embedding.

Order isomorphism preserves properties of posets, such as order, joins and meets, etc.

Definition 120. *Antitone order isomorphism* is a surjective antitone order embedding.

Definition 121.

1°. *Join semilattice homomorphism* is a function f from a join semilattice $\mathfrak{A}$ to a join semilattice $\mathfrak{B}$, such that $f(x \sqcup y) = fx \sqcup fy$ for every $x, y \in \mathfrak{A}$.
2°. *Meet semilattice homomorphism* is a function f from a meet semilattice $\mathfrak{A}$ to a meet semilattice $\mathfrak{B}$, such that $f(x \sqcap y) = fx \sqcap fy$ for every $x, y \in \mathfrak{A}$.

Obvious 122.

1°. Join semilattice homomorphisms are monotone.
2°. Meet semilattice homomorphisms are monotone.

Definition 123. A *lattice homomorphism* is a function from a lattice to a lattice, which is both join semilattice homomorphism and meet semilattice homomorphism.

Definition 124. *Complete lattice homomorphism* from a complete lattice $\mathfrak{A}$ to a complete lattice $\mathfrak{B}$ is a function f from $\mathfrak{A}$ to $\mathfrak{B}$ which preserves all meets and joins, that is $f \bigsqcup S = \bigsqcup \langle f \rangle^* S$ and $f \bigsqcap S = \bigsqcap \langle f \rangle^* S$ for every $S \in \mathscr{P}\mathfrak{A}$.

2.1.13. Galois connections. See [3, 12] for more detailed treatment of Galois connections.

Definition 125. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two posets. A *Galois connection* between $\mathfrak{A}$ and $\mathfrak{B}$ is a pair of functions $f = (f^*, f_*)$ with $f^* : \mathfrak{A} \to \mathfrak{B}$ and $f_* : \mathfrak{B} \to \mathfrak{A}$ such that:

$$\forall x \in \mathfrak{A}, y \in \mathfrak{B} : (f^* x \sqsubseteq y \Leftrightarrow x \sqsubseteq f_* y).$$

f_* is called *the upper adjoint* of f^* and f^* is called *the lower adjoint* of f_*.

Theorem 126. A pair (f^*, f_*) of functions $f^* : \mathfrak{A} \to \mathfrak{B}$ and $f_* : \mathfrak{B} \to \mathfrak{A}$ is a Galois connection iff both of the following:

1°. f^* and f_* are monotone.
2°. $x \sqsubseteq f_* f^* x$ and $f^* f_* y \sqsubseteq y$ for every $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$.

PROOF.

$\Rightarrow$.

2°. $x \sqsubseteq f_* f^* x$ since $f^* x \sqsubseteq f^* x$; $f^* f_* y \sqsubseteq y$ since $f_* y \sqsubseteq f_* y$.

1°. Let $a, b \in \mathfrak{A}$ and $a \sqsubseteq b$. Then $a \sqsubseteq b \sqsubseteq f_* f^* b$. So by definition $f^* a \sqsubseteq f^* b$ that is f^* is monotone. Analogously f_* is monotone.

$\Leftarrow$. $f^* x \sqsubseteq y \Rightarrow f_* f^* x \sqsubseteq f_* y \Rightarrow x \sqsubseteq f_* y$. The other direction is analogous.

□

THEOREM 127.

1°. $f^* \circ f_* \circ f^* = f^*$.
2°. $f_* \circ f^* \circ f_* = f_*$.

PROOF.

1°. Let $x \in \mathfrak{A}$. We have $x \sqsubseteq f_* f^* x$; consequently $f^* x \sqsubseteq f^* f_* f^* x$. On the other hand, $f^* f_* f^* x \sqsubseteq f^* x$. So $f^* f_* f^* x = f^* x$.

2°. Similar.

□

DEFINITION 128. A function f is called *idempotent* iff $f(f(X)) = f(X)$ for every argument X.

PROPOSITION 129. $f^* \circ f_*$ and $f_* \circ f^*$ are idempotent.

PROOF. $f^* \circ f_*$ is idempotent because $f^* f_* f^* f_* y = f^* f_* y$. $f_* \circ f^*$ is similar. □

THEOREM 130. Each of two adjoints is uniquely determined by the other.

PROOF. Let p and q be both upper adjoints of f. We have for all $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$:

$$x \sqsubseteq p(y) \Leftrightarrow f(x) \sqsubseteq y \Leftrightarrow x \sqsubseteq q(y).$$

For $x = p(y)$ we obtain $p(y) \sqsubseteq q(y)$ and for $x = q(y)$ we obtain $q(y) \sqsubseteq p(y)$. So $q(y) = p(y)$. □

THEOREM 131. Let f be a function from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.

1°. Both:
   (a) If f is monotone and $g(b) = \max\left\{\frac{x \in \mathfrak{A}}{fx \sqsubseteq b}\right\}$ is defined for every $b \in \mathfrak{B}$ then g is the upper adjoint of f.
   (b) If $g : \mathfrak{B} \to \mathfrak{A}$ is the upper adjoint of f then $g(b) = \max\left\{\frac{x \in \mathfrak{A}}{fx \sqsubseteq b}\right\}$ for every $b \in \mathfrak{B}$.

2°. Both:
   (a) If f is monotone and $g(b) = \min\left\{\frac{x \in \mathfrak{A}}{fx \sqsupseteq b}\right\}$ is defined for every $b \in \mathfrak{B}$ then g is the lower adjoint of f.
   (b) If $g : \mathfrak{B} \to \mathfrak{A}$ is the lower adjoint of f then $g(b) = \min\left\{\frac{x \in \mathfrak{A}}{fx \sqsupseteq b}\right\}$ for every $b \in \mathfrak{B}$.

PROOF. We will prove only the first as the second is its dual.

1°a. Let $g(b) = \max\left\{\frac{x \in \mathfrak{A}}{fx \sqsubseteq b}\right\}$ for every $b \in \mathfrak{B}$. Then

$$x \sqsubseteq gy \Leftrightarrow x \sqsubseteq \max\left\{\frac{x \in \mathfrak{A}}{fx \sqsubseteq y}\right\} \Rightarrow fx \sqsubseteq y$$

(because f is monotone) and

$$x \sqsubseteq gy \Leftrightarrow x \sqsubseteq \max\left\{\frac{x \in \mathfrak{A}}{fx \sqsubseteq y}\right\} \Leftarrow fx \sqsubseteq y.$$

So $fx \sqsubseteq y \Leftrightarrow x \sqsubseteq gy$ that is f is the lower adjoint of g.

1°b. We have

$$g(b) = \max\left\{\frac{x \in \mathfrak{A}}{fx \sqsubseteq b}\right\} \Leftrightarrow fgb \sqsubseteq b \wedge \forall x \in \mathfrak{A} : (fx \sqsubseteq b \Rightarrow x \sqsubseteq gb).$$

what is true by properties of adjoints.

□

THEOREM 132. Let f be a function from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.

1°. If f is an upper adjoint, f preserves all existing infima in $\mathfrak{A}$.
2°. If $\mathfrak{A}$ is a complete lattice and f preserves all infima, then f is an upper adjoint of a function $\mathfrak{B} \to \mathfrak{A}$.
3°. If f is a lower adjoint, f preserves all existing suprema in $\mathfrak{A}$.
4°. If $\mathfrak{A}$ is a complete lattice and f preserves all suprema, then f is a lower adjoint of a function $\mathfrak{B} \to \mathfrak{A}$.

PROOF. We will prove only first two items because the rest items are similar.

1°. Let $S \in \mathscr{P}\mathfrak{A}$ and $\bigsqcap S$ exists. $f\bigsqcap S$ is a lower bound for $\langle f\rangle^* S$ because f is order-preserving. If a is a lower bound for $\langle f\rangle^* S$ then $\forall x \in S : a \sqsubseteq fx$ that is $\forall x \in S : ga \sqsubseteq x$ where g is the lower adjoint of f. Thus $ga \sqsubseteq \bigsqcap S$ and hence $f\bigsqcap S \sqsupseteq a$. So $f\bigsqcap S$ is the greatest lower bound for $\langle f\rangle^* S$.

2°. Let $\mathfrak{A}$ be a complete lattice and f preserves all infima. Let

$$g(a) = \bigsqcap\left\{\frac{x \in \mathfrak{A}}{fx \sqsupseteq a}\right\}.$$

Since f preserves infima, we have

$$f(g(a)) = \bigsqcap\left\{\frac{f(x)}{x \in \mathfrak{A}, fx \sqsupseteq a}\right\} \sqsupseteq a.$$

$g(f(b)) = \bigsqcap\left\{\frac{x\in\mathfrak{A}}{fx\sqsupseteq fb}\right\} \sqsubseteq b.$

Obviously f is monotone and thus g is also monotone.

So f is the upper adjoint of g.

□

COROLLARY 133. Let f be a function from a complete lattice $\mathfrak{A}$ to a poset $\mathfrak{B}$. Then:

1°. f is an upper adjoint of a function $\mathfrak{B} \to \mathfrak{A}$ iff f preserves all infima in $\mathfrak{A}$.
2°. f is a lower adjoint of a function $\mathfrak{B} \to \mathfrak{A}$ iff f preserves all suprema in $\mathfrak{A}$.

2.1.13.1. *Order and composition of Galois connections.* Following [35] we will denote the set of Galois connection between posets $\mathfrak{A}$ and $\mathfrak{B}$ as $\mathfrak{A} \otimes \mathfrak{B}$.

DEFINITION 134. I will order Galois connections by the formula: $f \sqsubseteq g \Leftrightarrow f^* \sqsubseteq g^*$ (where $f^* \sqsubseteq g^* \Leftrightarrow \forall x \in \mathfrak{A} : f^*x \sqsubseteq g^*x$).

OBVIOUS 135. Galois connections $\mathfrak{A}\otimes\mathfrak{B}$ between two given posets form a poset.

PROPOSITION 136. $f \sqsubseteq g \Leftrightarrow f_* \sqsupseteq g_*$.

PROOF. It is enough to prove $f \sqsubseteq g \Rightarrow f_* \sqsupseteq g_*$ (the rest follows from the fact that a Galois connection is determined by one adjoint).

Really, let $f \sqsubseteq g$. Then $f_0^* \sqsubseteq f_1^*$ and thus:

$f_{0*}(b) = \max\left\{ \frac{x \in \mathfrak{A}}{f_0^* x \sqsubseteq b} \right\}$, $f_{1*}(b) = \max\left\{ \frac{x \in \mathfrak{A}}{f_1^* x \sqsubseteq b} \right\}$.

Thus $f_{0*}(b) \sqsupseteq f_{1*}(b)$ for every $b \in \mathfrak{B}$ and so $f_{0*} \sqsupseteq f_{1*}$. □

DEFINITION 137. Composition of Galois connections is defined by the formula: $g \circ f = (g^* \circ f^*, f_* \circ g_*)$.

PROPOSITION 138. Composition of Galois connections is a Galois connection.

PROOF. $g^* \circ f^*$ and $f_* \circ g_*$ are monotone as composition of monotone functions;

$(g^* \circ f^*)x \sqsubseteq z \Leftrightarrow g^* f^* x \sqsubseteq z \Leftrightarrow f^* x \sqsubseteq g_* z \Leftrightarrow x \sqsubseteq f_* g_* z \Leftrightarrow x \sqsubseteq (f_* \circ g_*)z$.

□

OBVIOUS 139. Composition of Galois connections preserves order.

2.1.13.2. *Antitone Galois connections.*

DEFINITION 140. An *antitone Galois connection* between posets $\mathfrak{A}$ and $\mathfrak{B}$ is a Galois connection between $\mathfrak{A}$ and $\operatorname{dual} \mathfrak{B}$.

OBVIOUS 141. An antitone Galois connection is a pair of antitone functions $f : \mathfrak{A} \to \mathfrak{B}$, $g : \mathfrak{B} \to \mathfrak{A}$ such that $b \sqsubseteq fa \Leftrightarrow a \sqsubseteq gb$ for every $a \in \mathfrak{A}$, $b \in \mathfrak{B}$.

Such f and g are called *polarities* (between $\mathfrak{A}$ and $\mathfrak{B}$).

OBVIOUS 142. $f \bigsqcup S = \bigsqcap \langle f \rangle^* S$ if f is a polarity between $\mathfrak{A}$ and $\mathfrak{B}$ and $S \in \mathscr{P}\mathfrak{A}$.

Galois connections (particularly between boolean lattices) are studied in [35] and [36].

2.1.14. Co-Brouwerian lattices.

DEFINITION 143. Let $\mathfrak{A}$ be a poset. *Pseudocomplement* of $a \in \mathfrak{A}$ is

$$\max\left\{ \frac{c \in \mathfrak{A}}{c \asymp a} \right\}.$$

If z is the pseudocomplement of a we will denote $z = a^*$.

DEFINITION 144. Let $\mathfrak{A}$ be a poset. *Dual pseudocomplement* of $a \in \mathfrak{A}$ is

$$\min\left\{ \frac{c \in \mathfrak{A}}{c \equiv a} \right\}.$$

If z is the dual pseudocomplement of a we will denote $z = a^+$.

PROPOSITION 145. If a is a complemented element of a bounded distributive lattice, then $\bar{a}$ is both pseudocomplement and dual pseudocomplement of a.

PROOF. Because of duality it is enough to prove that $\bar{a}$ is pseudocomplement of a.

We need to prove $c \asymp a \Rightarrow c \sqsubseteq \bar{a}$ for every element c of our poset, and $\bar{a} \asymp a$. The second is obvious. Let's prove $c \asymp a \Rightarrow c \sqsubseteq \bar{a}$.

Really, let $c \asymp a$. Then $c \sqcap a = \bot$; $\bar{a} \sqcup (c \sqcap a) = \bar{a}$; $(\bar{a} \sqcup c) \sqcap (\bar{a} \sqcup a) = \bar{a}$; $\bar{a} \sqcup c = \bar{a}$; $c \sqsubseteq \bar{a}$. □

DEFINITION 146. Let $\mathfrak{A}$ be a join-semilattice. Let $a, b \in \mathfrak{A}$. *Pseudodifference* of a and b is

$$\min\left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}.$$

If z is a pseudodifference of a and b we will denote $z = a \setminus^* b$.

REMARK 147. I do not require that a^* is undefined if there are no pseudocomplement of a and likewise for dual pseudocomplement and pseudodifference. In fact below I will define quasicomplement, dual quasicomplement, and quasidifference which generalize pseudo-* counterparts. I will denote a^* the more general case of quasicomplement than of pseudocomplement, and likewise for other notation.

OBVIOUS 148. Dual pseudocomplement is the dual of pseudocomplement.

THEOREM 149. Let $\mathfrak{A}$ be a distributive lattice with least element. Let $a, b \in \mathfrak{A}$. If $a \setminus b$ exists, then $a \setminus^* b$ also exists and $a \setminus^* b = a \setminus b$.

PROOF. Because $\mathfrak{A}$ be a distributive lattice with least element, the definition of $a \setminus b$ is correct.

Let $x = a \setminus b$ and let $S = \left\{ \frac{y \in \mathfrak{A}}{a \sqsubseteq b \sqcup y} \right\}$.

We need to show

1°. $x \in S$;

2°. $y \in S \Rightarrow x \sqsubseteq y$ (for every $y \in \mathfrak{A}$).

Really,

1°. Because $b \sqcup x = a \sqcup b$.

2°.

$$\begin{aligned}
& y \in S \\
\Rightarrow\ & a \sqsubseteq b \sqcup y && \text{(by definition of } S\text{)} \\
\Rightarrow\ & a \sqcup b \sqsubseteq b \sqcup y \\
\Rightarrow\ & x \sqcup b \sqsubseteq b \sqcup y && \text{(since } x \sqcup b = a \sqcup b\text{)} \\
\Rightarrow\ & x \sqcap (x \sqcup b) \sqsubseteq x \sqcap (b \sqcup y) \\
\Rightarrow\ & (x \sqcap x) \sqcup (x \sqcap b) \sqsubseteq (x \sqcap b) \sqcup (x \sqcap y) && \text{(by distributive law)} \\
\Rightarrow\ & x \sqcup \bot \sqsubseteq \bot \sqcup (x \sqcap y) && \text{(since } x \sqcap b = \bot\text{)} \\
\Rightarrow\ & x \sqsubseteq x \sqcap y \\
\Rightarrow\ & x \sqsubseteq y.
\end{aligned}$$

□

DEFINITION 150. *Co-brouwerian lattice* is a lattice for which pseudodifference of any two its elements is defined.

PROPOSITION 151. Every non-empty co-brouwerian lattice $\mathfrak{A}$ has least element.

PROOF. Let a be an arbitrary lattice element. Then

$$a \setminus^* a = \min \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq a \sqcup z} \right\} = \min \mathfrak{A}.$$

So $\min \mathfrak{A}$ exists. □

DEFINITION 152. *Co-Heyting lattice* is co-brouwerian lattice with greatest element.

DEFINITION 153. A *co-frame* is the same as a complete co-brouwerian lattice.

THEOREM 154. For a co-brouwerian lattice $a \sqcup -$ is an upper adjoint of $- \setminus^* a$ for every $a \in \mathfrak{A}$.

PROOF. $g(b) = \min \left\{ \frac{x \in \mathfrak{A}}{a \sqcup x \sqsupseteq b} \right\} = b \setminus^* a$ exists for every $b \in \mathfrak{A}$ and thus is the lower adjoint of $a \sqcup -$. □

COROLLARY 155. $\forall a, x, y \in \mathfrak{A} : (x \setminus^* a \sqsubseteq y \Leftrightarrow x \sqsubseteq a \sqcup y)$ for a co-brouwerian lattice.

Corollary 156. For a co-brouwerian lattice $a \sqcup \bigsqcap S = \bigsqcap \langle a \sqcup \rangle^* S$ whenever $\bigsqcap S$ exists (for a being a lattice element and S being a set of lattice elements).

Definition 157. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a complete lattice. *Quasidifference* $a \setminus^* b$ is defined by the formula:

$$a \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}.$$

Remark 158. A more detailed theory of quasidifference (as well as quasicomplement and dual quasicomplement) will be considered below.

Lemma 159. $(a \setminus^* b) \sqcup b = a \sqcup b$ for elements a, b of a meet infinite distributive complete lattice.

Proof.

$$\begin{aligned} (a \setminus^* b) \sqcup b &= \\ \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\} \sqcup b &= \\ \bigsqcap \left\{ \frac{z \sqcup b}{z \in \mathfrak{A}, a \sqsubseteq b \sqcup z} \right\} &= \\ \bigsqcap \left\{ \frac{t \in \mathfrak{A}}{t \sqsupseteq b, a \sqsubseteq t} \right\} &= \\ a \sqcup b. \end{aligned}$$

□

Theorem 160. The following are equivalent for a complete lattice $\mathfrak{A}$:

1°. $\mathfrak{A}$ is a co-frame.
2°. $\mathfrak{A}$ is meet infinite distributive.
3°. $\mathfrak{A}$ is a co-brouwerian lattice.
4°. $\mathfrak{A}$ is a co-Heyting lattice.
5°. $a \sqcup -$ has lower adjoint for every $a \in \mathfrak{A}$.

Proof. □

1°⇔3°. Because it is complete.
3°⇔4°. Obvious (taking into account completeness of $\mathfrak{A}$).
5°⇒2°. Let $- \setminus^* a$ be the lower adjoint of $a \sqcup -$. Let $S \in \mathscr{P}\mathfrak{A}$. For every $y \in S$ we have $y \sqsupseteq (a \sqcup y) \setminus^* a$ by properties of Galois connections; consequently $y \sqsupseteq \left(\bigsqcap \langle a \sqcup \rangle^* S\right) \setminus^* a$; $\bigsqcap S \sqsupseteq \left(\bigsqcap \langle a \sqcup \rangle^* S\right) \setminus^* a$. So

$$a \sqcup \bigsqcap S \sqsupseteq \left(\left(\bigsqcap \langle a \sqcup \rangle^* S \right) \setminus^* a \right) \sqcup a \sqsupseteq \bigsqcap \langle a \sqcup \rangle^* S.$$

But $a \sqcup \bigsqcap S \sqsubseteq \bigsqcap \langle a \sqcup \rangle^* S$ is obvious.
2°⇒3°. Let $a \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$. To prove that $\mathfrak{A}$ is a co-brouwerian lattice it is enough to prove $a \sqsubseteq b \sqcup (a \setminus^* b)$. But it follows from the lemma.
3°⇒5°. $a \setminus^* b = \min \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$. So $a \sqcup -$ is the upper adjoint of $- \setminus^* a$.
2°⇒5°. Because $a \sqcup -$ preserves all meets.

Corollary 161. Co-brouwerian lattices are distributive.

The following theorem is essentially borrowed from [19]:

Theorem 162. A lattice $\mathfrak{A}$ with least element $\bot$ is co-brouwerian with pseudodifference $\setminus^*$ iff $\setminus^*$ is a binary operation on $\mathfrak{A}$ satisfying the following identities:

1°. $a \setminus^* a = \bot$;

2°. $a \sqcup (b \setminus^* a) = a \sqcup b$;
3°. $b \sqcup (b \setminus^* a) = b$;
4°. $(b \sqcup c) \setminus^* a = (b \setminus^* a) \sqcup (c \setminus^* a)$.

PROOF.

$\Leftarrow$. We have

$$c \sqsupseteq b \setminus^* a \Rightarrow c \sqcup a \sqsupseteq a \sqcup (b \setminus^* a) = a \sqcup b \sqsupseteq b;$$

$$c \sqcup a \sqsupseteq b \Rightarrow c = c \sqcup (c \setminus^* a) \sqsupseteq (a \setminus^* a) \sqcup (c \setminus^* a) = (a \sqcup c) \setminus^* a \sqsupseteq b \setminus^* a.$$

So $c \sqsupseteq b \setminus^* a \Leftrightarrow c \sqcup a \sqsupseteq b$ that is $a \sqcup -$ is an upper adjoint of $- \setminus^* a$. By a theorem above our lattice is co-brouwerian. By another theorem above $\setminus^*$ is a pseudodifference.

$\Rightarrow$.

1°. Obvious.
2°.

$$\begin{aligned} a \sqcup (b \setminus^* a) &= \\ a \sqcup \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{b \sqsubseteq a \sqcup z} \right\} &= \\ \bigsqcap \left\{ \frac{a \sqcup z}{z \in \mathfrak{A}, b \sqsubseteq a \sqcup z} \right\} &= \\ a \sqcup b. \end{aligned}$$

3°. $b \sqcup (b \setminus^* a) = b \sqcup \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{b \sqsubseteq a \sqcup z} \right\} = \bigsqcap \left\{ \frac{b \sqcup z}{z \in \mathfrak{A}, b \sqsubseteq a \sqcup z} \right\} = b$.
4°. Obviously $(b \sqcup c) \setminus^* a \sqsupseteq b \setminus^* a$ and $(b \sqcup c) \setminus^* a \sqsupseteq c \setminus^* a$. Thus $(b \sqcup c) \setminus^* a \sqsupseteq (b \setminus^* a) \sqcup (c \setminus^* a)$. We have

$$\begin{aligned} (b \setminus^* a) \sqcup (c \setminus^* a) \sqcup a &= \\ ((b \setminus^* a) \sqcup a) \sqcup ((c \setminus^* a) \sqcup a) &= \\ (b \sqcup a) \sqcup (c \sqcup a) &= \\ a \sqcup b \sqcup c &\sqsupseteq \\ b \sqcup c. \end{aligned}$$

From this by definition of adjoints: $(b \setminus^* a) \sqcup (c \setminus^* a) \sqsupseteq (b \sqcup c) \setminus^* a$.

□

THEOREM 163. $(\bigsqcup S) \setminus^* a = \bigsqcup_{x \in S} (x \setminus^* a)$ for all $a \in \mathfrak{A}$ and $S \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a co-brouwerian lattice and $\bigsqcup S$ is defined.

PROOF. Because lower adjoint preserves all suprema. □

THEOREM 164. $(a \setminus^* b) \setminus^* c = a \setminus^* (b \sqcup c)$ for elements a, b, c of a co-frame.

PROOF. $a \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$.
$(a \setminus^* b) \setminus^* c = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \setminus^* b \sqsubseteq c \sqcup z} \right\}$.
$a \setminus^* (b \sqcup c) = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup c \sqcup z} \right\}$.
It is left to prove $a \setminus^* b \sqsubseteq c \sqcup z \Leftrightarrow a \sqsubseteq b \sqcup c \sqcup z$. But this follows from corollary 155. □

COROLLARY 165. $(((a_0 \setminus^* a_1) \setminus^* \dots) \setminus^* a_n) = a_0 \setminus^* (a_1 \sqcup \dots \sqcup a_n)$.

PROOF. By math induction. □

2.1.15. Dual pseudocomplement on co-Heyting lattices.

THEOREM 166. For co-Heyting algebras $\top \setminus^* b = b^+$.

PROOF.

$$\top \setminus^* b = \min\left\{ \frac{z \in \mathfrak{A}}{\top \sqsubseteq b \sqcup z} \right\} = \min\left\{ \frac{z \in \mathfrak{A}}{\top = b \sqcup z} \right\} = \min\left\{ \frac{z \in \mathfrak{A}}{b \equiv z} \right\} = b^+.$$

□

THEOREM 167. $(a \sqcap b)^+ = a^+ \sqcup b^+$ for every elements a, b of a co-Heyting algebra.

PROOF. $a \sqcup (a \sqcap b)^+ \sqsupseteq (a \sqcap b) \sqcup (a \sqcap b)^+ \sqsupseteq \top$. So $a \sqcup (a \sqcap b)^+ \sqsupseteq \top$; $(a \sqcap b)^+ \sqsupseteq \top \setminus^* a = a^+$.

We have $(a \sqcap b)^+ \sqsupseteq a^+$. Similarly $(a \sqcap b)^+ \sqsupseteq b^+$. Thus $(a \sqcap b)^+ \sqsupseteq a^+ \sqcup b^+$.

On the other hand, $a^+ \sqcup b^+ \sqcup (a \sqcap b) = (a^+ \sqcup b^+ \sqcup a) \sqcap (a^+ \sqcup b^+ \sqcup b)$. Obviously $a^+ \sqcup b^+ \sqcup a = a^+ \sqcup b^+ \sqcup b = \top$. So $a^+ \sqcup b^+ \sqcup (a \sqcap b) \sqsupseteq \top$ and thus $a^+ \sqcup b^+ \sqsupseteq \top \setminus^* (a \sqcap b) = (a \sqcap b)^+$.

So $(a \sqcap b)^+ = a^+ \sqcup b^+$. □

2.2. Intro to category theory

This is a *very* basic introduction to category theory.

DEFINITION 168. A *directed multigraph* (also known as *quiver*) is:

1°. a set $\mathcal{O}$ (*vertices*);
2°. a set $\mathcal{M}$ (*edges*);
3°. functions Src and Dst (*source* and *destination*) from $\mathcal{M}$ to $\mathcal{O}$.

Note that in category theory vertices are called *objects* and edges are called *morphisms*.

DEFINITION 169. A *semicategory* is a directed multigraph together with a partial binary operation $\circ$ on the set $\mathcal{M}$ such that $g \circ f$ is defined iff $\operatorname{Dst} f = \operatorname{Src} g$ (for every morphisms f and g) such that

1°. $\operatorname{Src}(g \circ f) = \operatorname{Src} f$ and $\operatorname{Dst}(g \circ f) = \operatorname{Dst} g$ whenever the composition $g \circ f$ of morphisms f and g is defined.
2°. $(h \circ g) \circ f = h \circ (g \circ f)$ whenever compositions in this equation are defined.

DEFINITION 170. The set $\operatorname{Hom}(A, B)$ (also denoted as $\operatorname{Hom}_C(A, B)$ or just $C(A, B)$, where C is our category) (morphisms from an object A to an object B) is exactly morphisms which have A as the source and B as the destination.

DEFINITION 171. *Identity morphism* is such a morphism e that $e \circ f = f$ and $g \circ e = g$ whenever compositions in these formulas are defined.

DEFINITION 172. A *category* is a semicategory with additional requirement that for every object X there exists identity morphism 1_X.

PROPOSITION 173. For every object X there exist no more than one identity morphism.

PROOF. Let p and q be both identity morphisms for a object X. Then $p = p \circ q = q$. □

DEFINITION 174. An *isomorphism* is such a morphism f of a category that there exists a morphism f^{-1} (*inverse* of f) such that $f \circ f^{-1} = 1_{\operatorname{Dst} f}$ and $f^{-1} \circ f = 1_{\operatorname{Src} f}$.

PROPOSITION 175. An isomorphism has exactly one inverse.

PROOF. Let g and h be both inverses of f. Then $h = h \circ 1_{\operatorname{Dst} f} = h \circ f \circ g = 1_{\operatorname{Src} f} \circ g = g$. $\square$

DEFINITION 176. A *groupoid* is a category all of whose morphisms are isomorphisms.

DEFINITION 177. A morphism whose source is the same as destination is called *endomorphism*.

DEFINITION 178. An *involution* or *involutive morphism* is an endomorphism f that $f \circ f = 1_{\operatorname{Ob} f}$. In other words, an involution is such a self-inverse (that is conforming to the formula $f = f^{-1}$) isomorphism.

DEFINITION 179. *Functor* from category C to category D is a mapping F which associates every object X of C with an object $F(X)$ of D and every morphism $f : X \to Y$ of C with morphism $F(f) : F(X) \to F(Y)$ of D, such that:

1°. $F(g \circ f) = F(g) \circ F(f)$ for every composable morphisms f, g of C;
2°. $F(1_X^C) = 1_{FX}^D$ for every object X of C.

DEFINITION 180. *Opposite category* C^{op} of category C is the category where "all arrows are reversed" that is every morphism f is replaced with so called *opposite morphism* f^{op} such that $\operatorname{Src} f^{\mathrm{op}} = \operatorname{Dst} f$, $\operatorname{Dst} f^{\mathrm{op}} = \operatorname{Src} f$ and $g^{\mathrm{op}} \circ f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}$ (composition taken in the "opposite order").

DEFINITION 181. A *monomorphism* (also called a *monic* morphism or a *mono*) is a left-cancellative morphism. That is, an arrow $f\colon X \to Y$ such that for all objects Z and all morphisms $g_1, g_2\colon Z \to X$,

$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2.$$

Monomorphisms are a categorical generalization of injective functions (also called "one-to-one functions"); in some categories the notions coincide, but monomorphisms are more general.

DEFINITION 182. The categorical dual of a monomorphism is an *epimorphism*, i.e. a monomorphism in a category C is an epimorphism in the dual category C^{op} that is it conforms to the formula

$$g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2.$$

2.2.1. Some important examples of categories.

EXERCISE 183. Prove that the below examples of categories are really categories.

DEFINITION 184. The category **Set** is:

- Objects are small sets.
- Morphisms from an object A to an object B are triples (A, B, f) where f is a function from A to B.
- Composition of morphisms is defined by the formula: $(B, C, g) \circ (A, B, f) = (A, C, g \circ f)$ where $g \circ f$ is function composition.

DEFINITION 185. The category **Rel** is:

- Objects are small sets.
- Morphisms from an object A to an object B are triples (A, B, f) where f is a binary relation between A and B.
- Composition of morphisms is defined by the formula: $(B, C, g) \circ (A, B, f) = (A, C, g \circ f)$ where $g \circ f$ is relation composition.

I will denote $\operatorname{GR}(A, B, f) = f$ for any morphism (A, B, f) of either **Set** or **Rel**.

Definition 186. A *subcategory* of a category C is a category whose set of objects is a subset of the set of objects of C and whose set of morphisms is a subset of the set of morphisms of C.

Definition 187. *Wide subcategory* of a category $(\mathcal{O}, \mathcal{M})$ is a category $(\mathcal{O}, \mathcal{M}')$ where $\mathcal{M} \subseteq \mathcal{M}'$ and the composition on $(\mathcal{O}, \mathcal{M}')$ is a restriction of composition of $(\mathcal{O}, \mathcal{M})$. (Similarly *wide sub-semicategory* can be defined.)

2.2.2. Commutative diagrams.

Definition 188. A *finite path in directed multigraph* is a tuple $[\![e_0, \dots, e_n]\!]$ of edges (where $i \in \mathbb{N}$) such that $\operatorname{Dst} e_i = \operatorname{Src} e_{i+1}$ for every $i = 0, \dots, n-1$.

Definition 189. The vertices of a finite path are $\operatorname{Src} e_0$, $\operatorname{Dst} e_0 = \operatorname{Src} e_1$, $\operatorname{Dst} e_1 = \operatorname{Src} e_2$, $\dots$, $\operatorname{Dst} e_n$.

Definition 190. Composition of finite paths $[\![e_0, \dots, e_n]\!]$ and $[\![e_k, \dots, e_m]\!]$ (where $\operatorname{Dst} e_n = \operatorname{Src} e_k$) is the path $[\![e_0, \dots, e_n, e_k, \dots e_m]\!]$. (It is a path because $\operatorname{Dst} e_n = \operatorname{Src} e_k$.)

Definition 191. A *cycle* is a finite path whose first vertex is the same as the last vertex (in other words $\operatorname{Dst} e_n = \operatorname{Src} e_0$).

Definition 192. A *diagram* in C is a directed multigraph, whose vertices are labeled with objects of C and whose edges are labeled with morphisms of C.

I will denote the morphism corresponding to a edge e as $D(e)$.

Definition 193. A diagram in C is *commutative* when the composition of morphisms corresponding to a finite path is always the same for finite paths from a fixed vertex A to a fixed vertex B independently of the path choice.

We will say "*commutative diagram*" when commutativity of a diagram is implied by the context.

Remark 194. See Wikipedia for more on definition and examples of commutative diagrams.

The following is an example of a commutative diagram in **Set** (because $x+5-3 = x+4-2$):

$$\begin{tikzcd} \mathbb{N} \arrow[r, "+5"] \arrow[d, "+4"] & \mathbb{N} \arrow[d, "-3"] \\ \mathbb{N} \arrow[r, "-2"] & \mathbb{N} \end{tikzcd}$$

We are especially interested in the special case of commutative diagrams every morphism of which is an isomorphism. So, the below theorem.

Theorem 195. If morphisms corresponding to every edge e_i of a cycle $[\![e_0, \dots, e_n]\!]$ are isomorphisms then the following are equivalent:

- The morphism induced by $[\![e_0, \dots, e_n]\!]$ is identity.
- The morphism induced by $[\![e_n, e_0, \dots, e_{n-1}]\!]$ is identity.
- The morphism induced by $[\![e_{n-1}, e_n, e_0, \dots, e_{n-2}]\!]$ is identity.
- $\dots$
- The morphism induced by $[\![e_1, e_2, \dots, e_n, e_0]\!]$ is identity.

In other words, the cycle being an identity does not depend on the choice of the start edge in the cycle.

Proof. Each step in the proof is like:

$$D(n) \circ \dots \circ D(e_0) = 1_{\operatorname{Src} D(e_0)} \Leftrightarrow$$
$$D(n)^{-1} \circ D(n) \circ \dots \circ D(e_0) \circ D(n) = D(n)^{-1} \circ 1_{\operatorname{Src} D(e_0)} \circ D(n) \Leftrightarrow$$
$$D(n-1) \circ \dots \circ D(e_0) \circ D(n) = 1_{\operatorname{Src} D(e_n)}.$$

□

LEMMA 196. Let f, g, h be isomorphisms. Let $g \circ f = h^{-1}$. The diagram at the figure 1 is commutative, every cycle in the diagram is an identity.

$$\begin{tikzcd}
 & \bullet \arrow[dl, shift right, "h"'] \arrow[dr, shift right, "f"'] & \\
\bullet \arrow[ur, shift right, "h^{-1}"'] \arrow[rr, shift left, "g^{-1}"] & & \bullet \arrow[ul, shift right, "f^{-1}"'] \arrow[ll, shift left, "g"]
\end{tikzcd}$$

FIGURE 1

PROOF. We will prove by induction that every cycle of the length N in the diagram is an identity.

For cycles of length 2 it holds by definition of isomorphism.

For cycles of length 3 it holds by theorem 195.

Consider a cycle of length above 3. It is easy to show that this cycle contains a sub-cycle of length 3 or below. (Consider three first edges $a \xrightarrow{e_0} b \xrightarrow{e_1} c \xrightarrow{e_2} d$ of the path, by pigeonhole principle we have that there are equal elements among a, b, c, d.) We can exclude the sub-cycle because it is identity. Thus we reduce to cycles of lesser length. Applying math induction, we get that every cycle in the diagram is an identity.

That the diagram is commutative follows from it (because for paths σ, τ we have the paths $\sigma \circ \tau^{-1}$ and $\tau \circ \sigma^{-1}$ being identities). □

LEMMA 197. Let f, g, h, t be isomorphisms. Let $t \circ h \circ g \circ f = 1_{\operatorname{Src} f}$. The diagram at the figure 2 is commutative, every cycle in the diagram is an identity.

$$\begin{tikzcd}
(0,0) \arrow[r, shift left, "f"] \arrow[d, shift left, "t^{-1}"] & (0,1) \arrow[l, shift left, "f^{-1}"] \arrow[d, shift left, "g"] \\
(1,0) \arrow[r, shift left, "h^{-1}"] \arrow[u, shift left, "t"] & (1,1) \arrow[l, shift left, "h"] \arrow[u, shift left, "g^{-1}"]
\end{tikzcd}$$

FIGURE 2

PROOF. Assign to every vertex (i, j) of the diagram morphism $W(i, j)$ defined by the table 1.

It is easy to verify by induction that the morphism corresponding to every path in the diagram starting at the vertex $(0, 0)$ and ending with a vertex (x, y) is $W(x, y)$.

Thus the morphism corresponding to every cycle starting at the vertex $(0, 0)$ is identity.

By symmetry, the morphism corresponding to every cycle is identity.

Table 1

i	j	$W(i,j)$
0	0	$1_{\operatorname{Src} f}$
0	1	f
1	0	t^{-1}
1	1	$g \circ f$

That the diagram is commutative follows from it (because for paths σ, τ we have the paths $\sigma \circ \tau^{-1}$ and $\tau \circ \sigma^{-1}$ being identities). □

2.3. Intro to group theory

Definition 198. A *semigroup* is a pair of a set G and an associative binary operation on G.

Definition 199. A *group* is a pair of a set G and a binary operation $\cdot$ on G such that:

1°. $(h \cdot g) \cdot f = h \cdot (g \cdot f)$ for every $f, g, h \in G$.
2°. There exists an element e (*identity*) of G such that $f \cdot e = e \cdot f = f$ for every $f \in G$.
3°. For every element f there exists an element f^{-1} (*inverse* of f) such that $f \cdot f^{-1} = f^{-1} \cdot f = e$.

Obvious 200. Every group is a semigroup.

Proposition 201. In every group there exists exactly one identity element.

Proof. If p and q are both identities, then $p = p \cdot q = q$. □

Proposition 202. Every group element has exactly one inverse.

Proof. Let p and q be both inverses of $f \in G$. Then $f \cdot p = p \cdot f = e$ and $f \cdot q = q \cdot f = e$. Then $p = p \cdot e = p \cdot f \cdot q = e \cdot q = q$. □

Proposition 203. $(g \cdot f)^{-1} = f^{-1} \cdot g^{-1}$ for every group elements f and g.

Proof. $(f^{-1} \cdot g^{-1}) \cdot (g \cdot f) = f^{-1} \cdot g^{-1} \cdot g \cdot f = f^{-1} \cdot e \cdot f = f^{-1} \cdot f = e$. Similarly $(g \cdot f) \cdot (f^{-1} \cdot g^{-1}) = e$. So $f^{-1} \cdot g^{-1}$ is the inverse of $g \cdot f$. □

Definition 204. A *permutation group* on a set D is a group whose elements are functions on D and whose composition is function composition.

Obvious 205. Elements of a permutation group are bijections.

Definition 206. A *transitive* permutation group on a set D is such a permutation group G on D that for every $x, y \in D$ there exists $r \in G$ such that $y = r(x)$.

A groupoid with single (arbitrarily chosen) object corresponds to every group. The morphisms of this category are elements of the group and the composition of morphisms is the group operation.

CHAPTER 3

More on order theory

3.1. Straight maps and separation subsets

3.1.1. Straight maps.

DEFINITION 207. An *order reflecting* map from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ is such a function f that (for every $x, y \in \mathfrak{A}$)

$$fx \sqsubseteq fy \Rightarrow x \sqsubseteq y.$$

OBVIOUS 208. Order embeddings are exactly the same as monotone and order reflecting maps.

DEFINITION 209. Let f be a monotone map from a meet-semilattice $\mathfrak{A}$ to some poset $\mathfrak{B}$. I call f a *straight* map when

$$\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow fa = f(a \sqcap b)).$$

PROPOSITION 210. The following statements are equivalent for a monotone map f:

1°. f is a straight map.
2°. $\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow fa \sqsubseteq f(a \sqcap b))$.
3°. $\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow fa \not\sqsupset f(a \sqcap b))$.
4°. $\forall a, b \in \mathfrak{A} : (fa \sqsupset f(a \sqcap b) \Rightarrow fa \not\sqsubseteq fb)$.

PROOF.
1°⇔2°⇔3°. Due $fa \sqsupseteq f(a \sqcap b)$.
3°⇔4°. Obvious.

□

REMARK 211. The definition of straight map can be generalized for any poset $\mathfrak{A}$ by the formula

$$\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow \exists c \in \mathfrak{A} : (c \sqsubseteq a \wedge c \sqsubseteq b \wedge fa = fc)).$$

This generalization is not yet researched however.

PROPOSITION 212. Let f be a monotone map from a meet-semilattice $\mathfrak{A}$ to a meet-semilattice $\mathfrak{B}$. If

$$\forall a, b \in \mathfrak{A} : f(a \sqcap b) = fa \sqcap fb$$

then f is a straight map.

PROOF. Let $fa \sqsubseteq fb$. Then $f(a \sqcap b) = fa \sqcap fb = fa$. □

PROPOSITION 213. Let f be a monotone map from a meet-semilattice $\mathfrak{A}$ to some poset $\mathfrak{B}$. If f is order reflecting, then f is a straight map.

PROOF. $fa \sqsubseteq fb \Rightarrow a \sqsubseteq b \Rightarrow a = a \sqcap b \Rightarrow fa = f(a \sqcap b)$. □

The following theorem is the main reason of why we are interested in straight maps:

THEOREM 214. If f is a straight monotone map from a meet-semilattice $\mathfrak{A}$ then the following statements are equivalent:

1°. f is an injection.
2°. f is order reflecting.
3°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow fa \sqsubset fb)$.
4°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow fa \neq fb)$.
5°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow fa \not\sqsupseteq fb)$.
6°. $\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow a \not\sqsupset b)$.

PROOF.

1°⇒3°. Let $a, b \in \mathfrak{A}$. Let $fa = fb \Rightarrow a = b$. Let $a \sqsubset b$. $fa \neq fb$ because $a \neq b$. $fa \sqsubseteq fb$ because $a \sqsubseteq b$. So $fa \sqsubset fb$.

2°⇒1°. Let $a, b \in \mathfrak{A}$. Let $fa \sqsubseteq fb \Rightarrow a \sqsubseteq b$. Let $fa = fb$. Then $a \sqsubseteq b$ and $b \sqsubseteq a$ and consequently $a = b$.

3°⇒2°. Let $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow fa \sqsubset fb)$. Let $a \not\sqsubseteq b$. Then $a \sqsupset a \sqcap b$. So $fa \sqsupset f(a \sqcap b)$. If $fa \sqsubseteq fb$ then $fa \sqsubseteq f(a \sqcap b)$ what is a contradiction.

3°⇒5°⇒4°. Obvious.

4°⇒3°. Because $a \sqsubset b \Rightarrow a \sqsubseteq b \Rightarrow fa \sqsubseteq fb$.

5°⇔6°. Obvious.

□

3.1.2. Separation subsets and full stars.

DEFINITION 215. $\partial_Y a = \left\{ \frac{x \in Y}{x \not\sqsubset a} \right\}$ for an element a of a poset $\mathfrak{A}$ and $Y \in \mathscr{P}\mathfrak{A}$.

DEFINITION 216. *Full star* of $a \in \mathfrak{A}$ is $\star a = \partial_{\mathfrak{A}} a$.

PROPOSITION 217. If $\mathfrak{A}$ is a meet-semilattice, then $\star$ is a straight monotone map.

PROOF. Monotonicity is obvious. Let $\star a \not\sqsubseteq \star(a \sqcap b)$. Then it exists $x \in \star a$ such that $x \notin \star(a \sqcap b)$. So $x \sqcap a \notin \star b$ but $x \sqcap a \in \star a$ and consequently $\star a \not\sqsubseteq \star b$. □

DEFINITION 218. A *separation subset* of a poset $\mathfrak{A}$ is such its subset Y that

$$\forall a, b \in \mathfrak{A} : (\partial_Y a = \partial_Y b \Rightarrow a = b).$$

DEFINITION 219. I call *separable* such poset that $\star$ is an injection.

DEFINITION 220. I call *strongly separable* such poset that $\star$ is order reflecting.

OBVIOUS 221. A poset is separable iff it has a separation subset.

OBVIOUS 222. A poset is strongly separable iff $\star$ is order embedding.

OBVIOUS 223. Strong separability implies separability.

DEFINITION 224. A poset $\mathfrak{A}$ has *disjunction property of Wallman* iff for any $a, b \in \mathfrak{A}$ either $b \sqsubseteq a$ or there exists a non-least element $c \sqsubseteq b$ such that $a \asymp c$.

THEOREM 225. For a meet-semilattice with least element the following statements are equivalent:

1°. $\mathfrak{A}$ is separable.
2°. $\mathfrak{A}$ is strongly separable.
3°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \star a \sqsubset \star b)$.
4°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \star a \neq \star b)$.
5°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \star a \not\sqsupseteq \star b)$.
6°. $\forall a, b \in \mathfrak{A} : (\star a \sqsubseteq \star b \Rightarrow a \not\sqsupset b)$.
7°. $\mathfrak{A}$ conforms to Wallman's disjunction property.

8°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \exists c \in \mathfrak{A} \setminus \{\bot\} : (c \asymp a \wedge c \sqsubseteq b))$.

PROOF.

1°⇔2°⇔3°⇔4°⇔5°⇔6°. By the above theorem.

8°⇒4°. Let property 8° hold. Let $a \sqsubset b$. Then it exists element $c \sqsubseteq b$ such that $c \neq \bot$ and $c \sqcap a = \bot$. But $c \sqcap b \neq \bot$. So $\star a \neq \star b$.

2°⇒7°. Let property 2° hold. Let $a \not\sqsubseteq b$. Then $\star a \not\sqsubseteq \star b$ that is it there exists $c \in \star a$ such that $c \notin \star b$, in other words $c \sqcap a \neq \bot$ and $c \sqcap b = \bot$. Let $d = c \sqcap a$. Then $d \sqsubseteq a$ and $d \neq \bot$ and $d \sqcap b = \bot$. So disjunction property of Wallman holds.

7°⇒8°. Obvious.

8°⇒7°. Let $b \not\sqsubseteq a$. Then $a \sqcap b \sqsubset b$ that is $a' \sqsubset b$ where $a' = a \sqcap b$. Consequently $\exists c \in \mathfrak{A} \setminus \{\bot\} : (c \asymp a' \wedge c \sqsubseteq b)$. We have $c \sqcap a = c \sqcap b \sqcap a = c \sqcap a' = \bot$. So $c \sqsubseteq b$ and $c \sqcap a = \bot$. Thus Wallman's disjunction property holds.

□

PROPOSITION 226. Every boolean lattice is strongly separable.

PROOF. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a boolean lattice an $a \neq b$. Then $a \sqcap \bar{b} \neq \bot$ or $\bar{a} \sqcap b \neq \bot$ because otherwise $a \sqcap \bar{b} = \bot$ and $a \sqcup \bar{b} = \top$ and thus $a = b$. Without loss of generality assume $a \sqcap \bar{b} \neq \bot$. Then $a \sqcap c \neq \bot$ and $b \sqcap c = \bot$ for $c = a \sqcap \bar{b} \neq \bot$, that is our lattice is separable.

It is strongly separable by theorem 225. □

3.1.3. Atomically Separable Lattices.

PROPOSITION 227. "atoms" is a straight monotone map (for any meet-semilattice).

PROOF. Monotonicity is obvious. The rest follows from the formula

$$\operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$$

(corollary 109). □

DEFINITION 228. I will call *atomically separable* such a poset that "atoms" is an injection.

PROPOSITION 229. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \operatorname{atoms} a \subset \operatorname{atoms} b)$ iff $\mathfrak{A}$ is atomically separable for a poset $\mathfrak{A}$.

PROOF.

⇐. Obvious.

⇒. Let $a \neq b$ for example $a \not\sqsubseteq b$. Then $a \sqcap b \sqsubset a$; $\operatorname{atoms} a \supset \operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$ and thus $\operatorname{atoms} a \neq \operatorname{atoms} b$.

□

PROPOSITION 230. Any atomistic poset is atomically separable.

PROOF. We need to prove that $\operatorname{atoms} a = \operatorname{atoms} b \Rightarrow a = b$. But it is obvious because

$$a = \bigsqcup \operatorname{atoms} a \quad \text{and} \quad b = \bigsqcup \operatorname{atoms} b.$$

□

THEOREM 231. A complete lattice is atomistic iff it is atomically separable.

PROOF. Direct implication is the above proposition. Let's prove the reverse implication.

Let "atoms" be injective. Consider an element a of our poset. Let $b = \bigsqcup \operatorname{atoms} a$. Obviously $b \sqsubseteq a$ and thus $\operatorname{atoms} b \subseteq \operatorname{atoms} a$. But if $x \in \operatorname{atoms} a$ then $x \sqsubseteq b$ and thus $x \in \operatorname{atoms} b$. So $\operatorname{atoms} a = \operatorname{atoms} b$. By injectivity $a = b$ that is $a = \bigsqcup \operatorname{atoms} a$. □

THEOREM 232. If a lattice with least element is atomic and separable then it is atomistic.

PROOF. Suppose the contrary that is $a \sqsupset \bigsqcup \operatorname{atoms} a$. Then, because our lattice is separable, there exists $c \in \mathfrak{A}$ such that $c \sqcap a \neq \bot$ and $c \sqcap \bigsqcup \operatorname{atoms} a = \bot$. There exists atom $d \sqsubseteq c$ such that $d \sqsubseteq c \sqcap a$. $d \sqcap \bigsqcup \operatorname{atoms} a \sqsubseteq c \sqcap \bigsqcup \operatorname{atoms} a = \bot$. But $d \in \operatorname{atoms} a$. Contradiction. □

THEOREM 233. Let $\mathfrak{A}$ be an atomic meet-semilattice with least element. Then the following statements are equivalent:

1°. $\mathfrak{A}$ is separable.
2°. $\mathfrak{A}$ is strongly separable.
3°. $\mathfrak{A}$ is atomically separable.
4°. $\mathfrak{A}$ conforms to Wallman's disjunction property.
5°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \exists c \in \mathfrak{A} \setminus \{\bot\} : (c \asymp a \wedge c \sqsubseteq b))$.

PROOF.

1°⇔2°⇔4°⇔5°. Proved above.

3°⇒5°. Let our semilattice be atomically separable. Let $a \sqsubset b$. Then $\operatorname{atoms} a \subset \operatorname{atoms} b$ and there exists $c \in \operatorname{atoms} b$ such that $c \notin \operatorname{atoms} a$. $c \neq \bot$ and $c \sqsubseteq b$, from which (taking into account that c is an atom) $c \sqsubseteq b$ and $c \sqcap a = \bot$. So our semilattice conforms to the formula 5°.

5°⇒3°. Let formula 5° hold. Then for any elements $a \sqsubset b$ there exists $c \neq \bot$ such that $c \sqsubseteq b$ and $c \sqcap a = \bot$. Because $\mathfrak{A}$ is atomic there exists atom $d \sqsubseteq c$. $d \in \operatorname{atoms} b$ and $d \notin \operatorname{atoms} a$. So $\operatorname{atoms} a \neq \operatorname{atoms} b$ and $\operatorname{atoms} a \subseteq \operatorname{atoms} b$. Consequently $\operatorname{atoms} a \subset \operatorname{atoms} b$.

□

THEOREM 234. Any atomistic poset is strongly separable.

PROOF. $\star x \sqsubseteq \star y \Rightarrow \operatorname{atoms} x \sqsubseteq \operatorname{atoms} y \Rightarrow x \sqsubseteq y$ because $\operatorname{atoms} x = \star x \cap \operatorname{atoms}^{\mathfrak{A}}$. □

3.2. Quasidifference and Quasicomplement

I've got quasidifference and quasicomplement (and dual quasicomplement) replacing max and min in the definition of pseudodifference and pseudocomplement (and dual pseudocomplement) with $\bigsqcup$ and $\bigsqcap$. Thus quasidifference and (dual) quasicomplement are generalizations of their pseudo- counterparts.

REMARK 235. *Pseudocomplements* and *pseudodifferences* are standard terminology. *Quasi-* counterparts are my neologisms.

DEFINITION 236. Let $\mathfrak{A}$ be a poset, $a \in \mathfrak{A}$. *Quasicomplement* of a is

$$a^* = \bigsqcup \left\{ \frac{c \in \mathfrak{A}}{c \asymp a} \right\}.$$

DEFINITION 237. Let $\mathfrak{A}$ be a poset, $a \in \mathfrak{A}$. *Dual quasicomplement* of a is

$$a^+ = \bigsqcap \left\{ \frac{c \in \mathfrak{A}}{c \equiv a} \right\}.$$

I will denote quasicomplement and dual quasicomplement for a specific poset $\mathfrak{A}$ as $a^{*(\mathfrak{A})}$ and $a^{+(\mathfrak{A})}$.

Definition 238. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a distributive lattice. *Quasidifference* of a and b is

$$a \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}.$$

Definition 239. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a distributive lattice. *Second quasidifference* of a and b is

$$a \# b = \bigsqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \asymp b} \right\}.$$

Theorem 240. $a \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge a \sqsubseteq b \sqcup z} \right\}$ where $\mathfrak{A}$ is a distributive lattice and $a, b \in \mathfrak{A}$.

Proof. Obviously $\left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge a \sqsubseteq b \sqcup z} \right\} \subseteq \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$. Thus $\bigsqcap \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge a \sqsubseteq b \sqcup z} \right\} \sqsupseteq a \setminus^* b$.

Let $z \in \mathfrak{A}$ and $z' = z \sqcap a$.

$a \sqsubseteq b \sqcup z \Rightarrow a \sqsubseteq (b \sqcup z) \sqcap a \Leftrightarrow a \sqsubseteq (b \sqcap a) \sqcup (z \sqcap a) \Leftrightarrow a \sqsubseteq (b \sqcap a) \sqcup z' \Rightarrow a \sqsubseteq b \sqcup z'$ and $a \sqsubseteq b \sqcup z \Leftarrow a \sqsubseteq b \sqcup z'$. Thus $a \sqsubseteq b \sqcup z \Leftrightarrow a \sqsubseteq b \sqcup z'$.

If $z \in \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$ then $a \sqsubseteq b \sqcup z$ and thus

$$z' \in \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge a \sqsubseteq b \sqcup z} \right\}.$$

But $z' \sqsubseteq z$ thus having $\bigsqcap \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge a \sqsubseteq b \sqcup z} \right\} \sqsubseteq \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$. □

Remark 241. If we drop the requirement that $\mathfrak{A}$ is distributive, two formulas for quasidifference (the definition and the last theorem) fork.

Obvious 242. Dual quasicomplement is the dual of quasicomplement.

Obvious 243.

- Every pseudocomplement is quasicomplement.
- Every dual pseudocomplement is dual quasicomplement.
- Every pseudodifference is quasidifference.

Below we will stick to the more general quasies than pseudos. If needed, one can check that a quasicomplement a^* is a pseudocomplement by the equation $a^* \asymp a$ (and analogously with other quasies).

Next we will express quasidifference through quasicomplement.

Proposition 244.

1°. $a \setminus^* b = a \setminus^* (a \sqcap b)$ for any distributive lattice;
2°. $a \# b = a \#(a \sqcap b)$ for any distributive lattice with least element.

Proof.

1°. $a \sqsubseteq (a \sqcap b) \sqcup z \Leftrightarrow a \sqsubseteq (a \sqcup z) \sqcap (b \sqcup z) \Leftrightarrow a \sqsubseteq a \sqcup z \wedge a \sqsubseteq b \sqcup z \Leftrightarrow a \sqsubseteq b \sqcup z$. Thus $a \setminus^* (a \sqcap b) = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq (a \sqcap b) \sqcup z} \right\} = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\} = a \setminus^* b$.

2°.

$$\begin{aligned} a \# (a \sqcap b) &= \\ \bigsqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap a \sqcap b = \bot} \right\} &= \\ \bigsqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge (z \sqcap a) \sqcap a \sqcap b = \bot} \right\} &= \\ \bigsqcup \left\{ \frac{z \sqcap a}{z \in \mathfrak{A}, z \sqcap a \sqcap b = \bot} \right\} &= \\ \bigsqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a, z \sqcap b = \bot} \right\} &= \\ a \# b. \end{aligned}$$

□

I will denote Da the lattice $\left\{ \frac{x \in \mathfrak{A}}{x \sqsubseteq a} \right\}$.

THEOREM 245. For $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a distributive lattice

1°. $a \setminus^* b = (a \sqcap b)^{+(Da)}$;

2°. $a \# b = (a \sqcap b)^{*(Da)}$ if $\mathfrak{A}$ has least element.

PROOF.

1°.

$$\begin{aligned} (a \sqcap b)^{+(Da)} &= \\ \bigsqcap \left\{ \frac{c \in Da}{c \sqcup (a \sqcap b) = a} \right\} &= \\ \bigsqcap \left\{ \frac{c \in Da}{c \sqcup (a \sqcap b) \sqsupseteq a} \right\} &= \\ \bigsqcap \left\{ \frac{c \in Da}{(c \sqcup a) \sqcap (c \sqcup b) \sqsupseteq a} \right\} &= \\ \bigsqcap \left\{ \frac{c \in \mathfrak{A}}{c \sqsubseteq a \wedge c \sqcup b \sqsupseteq a} \right\} &= \\ a \setminus^* b. \end{aligned}$$

2°.

$$\begin{aligned} (a \sqcap b)^{*(Da)} &= \\ \bigsqcup \left\{ \frac{c \in Da}{c \sqcap a \sqcap b = \bot} \right\} &= \\ \bigsqcup \left\{ \frac{c \in \mathfrak{A}}{c \sqsubseteq a \wedge c \sqcap a \sqcap b = \bot} \right\} &= \\ \bigsqcup \left\{ \frac{c \in \mathfrak{A}}{c \sqsubseteq a \wedge c \sqcap b = \bot} \right\} &= \\ a \# b. \end{aligned}$$

□

PROPOSITION 246. $(a \sqcup b) \setminus^* b \sqsubseteq a$ for an arbitrary complete lattice.

PROOF. $(a \sqcup b) \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqcup b \sqsubseteq b \sqcup z} \right\}$.

But $a \sqsubseteq z \Rightarrow a \sqcup b \sqsubseteq b \sqcup z$. So $\left\{ \frac{z \in \mathfrak{A}}{a \sqcup b \sqsubseteq b \sqcup z} \right\} \supseteq \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq z} \right\}$.

Consequently, $(a \sqcup b) \setminus^* b \sqsubseteq \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq z} \right\} = a$. □

3.3. Several equal ways to express pseudodifference

THEOREM 247. For an atomistic co-brouwerian lattice $\mathfrak{A}$ and $a, b \in \mathfrak{A}$ the following expressions are always equal:

1°. $a \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$ (quasidifference of a and b);

2°. $a \# b = \bigsqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$ (second quasidifference of a and b);

3°. $\bigsqcup(\operatorname{atoms} a \setminus \operatorname{atoms} b)$.

PROOF.

Proof of 1°=3°.

$$\begin{aligned} a \setminus^* b &= \\ \left(\bigsqcup \operatorname{atoms} a\right) \setminus^* b &= \quad \text{(theorem 163)} \\ \bigsqcup_{A \in \operatorname{atoms} a} (A \setminus^* b) &= \\ \bigsqcup_{A \in \operatorname{atoms} a} \left(\begin{cases} A & \text{if } A \notin \operatorname{atoms} b \\ \bot & \text{if } A \in \operatorname{atoms} b \end{cases} \right) &= \\ \bigsqcup \left\{ \frac{A}{A \in \operatorname{atoms} a, A \notin \operatorname{atoms} b} \right\} &= \\ \bigsqcup(\operatorname{atoms} a \setminus \operatorname{atoms} b). \end{aligned}$$

Proof of 2°=3°. $a \setminus^* b$ is defined because our lattice is co-brouwerian. Taking the above into account, we have

$$\begin{aligned} a \setminus^* b &= \\ \bigsqcup(\operatorname{atoms} a \setminus \operatorname{atoms} b) &= \\ \bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot} \right\}. \end{aligned}$$

So $\bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot} \right\}$ is defined.

If $z \sqsubseteq a \wedge z \sqcap b = \bot$ then $z' = \bigsqcup \left\{ \frac{x \in \operatorname{atoms} z}{x \sqcap b = \bot} \right\}$ is defined because $z' = z \setminus^* b$ (atomisticity taken into account). z' is a lower bound for $\left\{ \frac{z \in \operatorname{atoms} a}{x \sqcap b = \bot} \right\}$.

Thus $z' \in \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$ and so $\bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot} \right\}$ is an upper bound of $\left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$.

If y is above every $z' \in \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$ then y is above every $z \in \operatorname{atoms} a$ such that $z \sqcap b = \bot$ and thus y is above $\bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot} \right\}$.

Thus $\bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot} \right\}$ is least upper bound of

$$\left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\},$$

that is

$$\begin{aligned} \bigsqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\} &= \\ \bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot} \right\} &= \\ \bigsqcup(\operatorname{atoms} a \setminus \operatorname{atoms} b). \end{aligned}$$

□

3.4. Partially ordered categories

3.4.1. Definition.

Definition 248. I will call a partially ordered (pre)category a (pre)category together with partial order $\sqsubseteq$ on each of its Mor-sets with the additional requirement that

$$f_1 \sqsubseteq f_2 \wedge g_1 \sqsubseteq g_2 \Rightarrow g_1 \circ f_1 \sqsubseteq g_2 \circ f_2$$

for every morphisms f_1, g_1, f_2, g_2 such that $\operatorname{Src} f_1 = \operatorname{Src} f_2$ and $\operatorname{Dst} f_1 = \operatorname{Dst} f_2 = \operatorname{Src} g_1 = \operatorname{Src} g_2$ and $\operatorname{Dst} g_1 = \operatorname{Dst} g_2$.

I will denote lattice operations on a Hom-set $C(A, B)$ of a category (or any directed multigraph) like $\sqcup^C$ instead of writing $\sqcup^{C(A,B)}$ explicitly.

3.4.2. Dagger categories.

Definition 249. I will call a *dagger semicategory* a semicategory together with an involutive contravariant identity-on-objects prefunctor $x \mapsto x^\dagger$.

In other words, a dagger semicategory is a semicategory equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g:

1°. $f^{\dagger\dagger} = f$;
2°. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.

Definition 250. I will call a *dagger category* a category together with an involutive contravariant identity-on-objects functor $x \mapsto x^\dagger$.

In other words, a dagger category is a category equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g and object A:

1°. $f^{\dagger\dagger} = f$;
2°. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$;
3°. $(1_A)^\dagger = 1_A$.

Theorem 251. If a category is a dagger semicategory then it is a dagger category.

Proof. We need to prove only that $(1_A)^\dagger = 1_A$. Really,

$$(1_A)^\dagger = (1_A)^\dagger \circ 1_A = (1_A)^\dagger \circ (1_A)^{\dagger\dagger} = ((1_A)^\dagger \circ 1_A)^\dagger = (1_A)^{\dagger\dagger} = 1_A.$$

□

For a partially ordered dagger (pre)category I will additionally require (for every morphisms f and g with the same source and destination)

$$f^\dagger \sqsubseteq g^\dagger \Leftrightarrow f \sqsubseteq g.$$

An example of dagger category is the category **Rel** whose objects are sets and whose morphisms are binary relations between these sets with usual composition of binary relations and with $f^\dagger = f^{-1}$.

Definition 252. A morphism f of a dagger category is called *unitary* when it is an isomorphism and $f^\dagger = f^{-1}$.

Definition 253. *Symmetric* (endo)morphism of a dagger semicategory is such a morphism f that $f = f^\dagger$.

Definition 254. *Transitive* (endo)morphism of a semicategory is such a morphism f that $f = f \circ f$.

THEOREM 255. The following conditions are equivalent for a morphism f of a dagger semicategory:

1°. f is symmetric and transitive.
2°. $f = f^\dagger \circ f$.

PROOF.

1°⇒2°. If f is symmetric and transitive then $f^\dagger \circ f = f \circ f = f$.
2°⇒1°. $f^\dagger = (f^\dagger \circ f)^\dagger = f^\dagger \circ f^{\dagger\dagger} = f^\dagger \circ f = f$, so f is symmetric. $f = f^\dagger \circ f = f \circ f$, so f is transitive.

□

3.4.2.1. *Some special classes of morphisms.*

DEFINITION 256. For a partially ordered dagger category I will call *monovalued* morphism such a morphism f that $f \circ f^\dagger \sqsubseteq 1_{\operatorname{Dst} f}$.

DEFINITION 257. For a partially ordered dagger category I will call *entirely defined* morphism such a morphism f that $f^\dagger \circ f \sqsupseteq 1_{\operatorname{Src} f}$.

DEFINITION 258. For a partially ordered dagger category I will call *injective* morphism such a morphism f that $f^\dagger \circ f \sqsubseteq 1_{\operatorname{Src} f}$.

DEFINITION 259. For a partially ordered dagger category I will call *surjective* morphism such a morphism f that $f \circ f^\dagger \sqsupseteq 1_{\operatorname{Dst} f}$.

REMARK 260. It is easy to show that this is a generalization of monovalued, entirely defined, injective, and surjective functions as morphisms of the category **Rel**.

OBVIOUS 261. "Injective morphism" is a dual of "monovalued morphism" and "surjective morphism" is a dual of "entirely defined morphism".

DEFINITION 262. For a given partially ordered dagger category C the *category of monovalued* (*entirely defined*, *injective*, *surjective*) morphisms of C is the category with the same set of objects as of C and the set of morphisms being the set of monovalued (entirely defined, injective, surjective) morphisms of C with the composition of morphisms the same as in C.

We need to prove that these are really categories, that is that composition of monovalued (entirely defined, injective, surjective) morphisms is monovalued (entirely defined, injective, surjective) and that identity morphisms are monovalued, entirely defined, injective, and surjective.

PROOF. We will prove only for monovalued morphisms and entirely defined morphisms, as injective and surjective morphisms are their duals.

Monovalued. Let f and g be monovalued morphisms, $\operatorname{Dst} f = \operatorname{Src} g$. Then

$$\begin{aligned}(g \circ f) \circ (g \circ f)^\dagger &= \\ g \circ f \circ f^\dagger \circ g^\dagger &\sqsubseteq \\ g \circ 1_{\operatorname{Src} g} \circ g^\dagger &= \\ g \circ g^\dagger &\sqsubseteq \\ 1_{\operatorname{Dst} g} &= 1_{\operatorname{Dst}(g \circ f)}.\end{aligned}$$

So $g \circ f$ is monovalued.

That identity morphisms are monovalued follows from the following:

$$1_A \circ (1_A)^\dagger = 1_A \circ 1_A = 1_A = 1_{\operatorname{Dst} 1_A} \sqsubseteq 1_{\operatorname{Dst} 1_A}.$$

Entirely defined. Let f and g be entirely defined morphisms, $\operatorname{Dst} f = \operatorname{Src} g$. Then

$$\begin{aligned}(g \circ f)^\dagger \circ (g \circ f) &= \\ f^\dagger \circ g^\dagger \circ g \circ f &\sqsupseteq \\ f^\dagger \circ 1_{\operatorname{Src} g} \circ f &= \\ f^\dagger \circ 1_{\operatorname{Dst} f} \circ f &= \\ f^\dagger \circ f &\sqsupseteq \\ 1_{\operatorname{Src} f} &= 1_{\operatorname{Src}(g \circ f)}.\end{aligned}$$

So $g \circ f$ is entirely defined.

That identity morphisms are entirely defined follows from the following:

$$(1_A)^\dagger \circ 1_A = 1_A \circ 1_A = 1_A = 1_{\operatorname{Src} 1_A} \sqsupseteq 1_{\operatorname{Src} 1_A}.$$

□

Definition 263. I will call a *bijective* morphism a morphism which is entirely defined, monovalued, injective, and surjective.

Proposition 264. If a morphism is bijective then it is an isomorphism.

Proof. Let f be bijective. Then $f \circ f^\dagger \sqsubseteq 1_{\operatorname{Dst} f}$, $f^\dagger \circ f \sqsupseteq 1_{\operatorname{Src} f}$, $f^\dagger \circ f \sqsubseteq 1_{\operatorname{Src} f}$, $f \circ f^\dagger \sqsupseteq 1_{\operatorname{Dst} f}$. Thus $f \circ f^\dagger = 1_{\operatorname{Dst} f}$ and $f^\dagger \circ f = 1_{\operatorname{Src} f}$ that is $f^\dagger$ is an inverse of f. □

Let Hom-sets be complete lattices.

Definition 265. A morphism f of a partially ordered category is *metamonovalued* when $(\bigsqcap G) \circ f = \bigsqcap_{g \in G}(g \circ f)$ whenever G is a set of morphisms with a suitable source and destination.

Definition 266. A morphism f of a partially ordered category is *metainjective* when $f \circ (\bigsqcap G) = \bigsqcap_{g \in G}(f \circ g)$ whenever G is a set of morphisms with a suitable source and destination.

Obvious 267. Metamonovaluedness and metainjectivity are dual to each other.

Definition 268. A morphism f of a partially ordered category is *metacomplete* when $f \circ (\bigsqcup G) = \bigsqcup_{g \in G}(f \circ g)$ whenever G is a set of morphisms with a suitable source and destination.

Definition 269. A morphism f of a partially ordered category is *cometacomplete* when $(\bigsqcup G) \circ f = \bigsqcup_{g \in G}(g \circ f)$ whenever G is a set of morphisms with a suitable source and destination.

Let now Hom-sets be meet-semilattices.

Definition 270. A morphism f of a partially ordered category is *weakly metamonovalued* when $(g \sqcap h) \circ f = (g \circ f) \sqcap (h \circ f)$ whenever g and h are morphisms with a suitable source and destination.

Definition 271. A morphism f of a partially ordered category is *weakly metainjective* when $f \circ (g \sqcap h) = (f \circ g) \sqcap (f \circ h)$ whenever g and h are morphisms with a suitable source and destination.

Let now Hom-sets be join-semilattices.

Definition 272. A morphism f of a partially ordered category is *weakly metacomplete* when $f \circ (g \sqcup h) = (f \circ g) \sqcup (f \circ h)$ whenever g and h are morphisms with a suitable source and destination.

Definition 273. A morphism f of a partially ordered category is *weakly co-metacomplete* when $(g \sqcup h) \circ f = (g \circ f) \sqcup (h \circ f)$ whenever g and h are morphisms with a suitable source and destination.

Obvious 274.

1°. Metamonovalued morphisms are weakly metamonovalued.
2°. Metainjective morphisms are weakly metainjective.
3°. Metacomplete morphisms are weakly metacomplete.
4°. Co-metacomplete morphisms are weakly co-metacomplete.

3.5. Partitioning

Definition 275. Let $\mathfrak{A}$ be a complete lattice. *Torning* of an element $a \in \mathfrak{A}$ is a set $S \in \mathscr{P}\mathfrak{A} \setminus \{\bot\}$ such that

$$\bigsqcup S = a \quad \text{and} \quad \forall x, y \in S : (x \neq y \Rightarrow x \asymp y).$$

Definition 276. Let $\mathfrak{A}$ be a complete lattice. *Weak partition* of an element $a \in \mathfrak{A}$ is a set $S \in \mathscr{P}\mathfrak{A} \setminus \{\bot\}$ such that

$$\bigsqcup S = a \quad \text{and} \quad \forall x \in S : x \asymp \bigsqcup (S \setminus \{x\}).$$

Definition 277. Let $\mathfrak{A}$ be a complete lattice. *Strong partition* of an element $a \in \mathfrak{A}$ is a set $S \in \mathscr{P}\mathfrak{A} \setminus \{\bot\}$ such that

$$\bigsqcup S = a \quad \text{and} \quad \forall A, B \in \mathscr{P}S : (A \asymp B \Rightarrow \bigsqcup A \asymp \bigsqcup B).$$

Obvious 278.

1°. Every strong partition is a weak partition.
2°. Every weak partition is a torning.

Definition 279. *Complete lattice generated by* a set P (on a complete lattice) is the set (obviously having the structure of complete lattice) $P_0 \cup P_1 \cup \dots$ where $P_0 = P$ and $P_{i+1} = \left\{ \frac{\bigsqcup K, \bigsqcap K}{K \in \mathscr{P}P_i} \right\}$.

Obvious 280. Complete lattice generated by a set is indeed a complete lattice.

Example 281. $[S] \neq \left\{ \frac{\bigsqcup^{\mathfrak{A}} X}{X \in \mathscr{P}S} \right\}$, where $[S]$ is the complete lattice generated by a strong partition S of a filter on a set.

Proof. Consider any infinite set U and its strong partition $S = \left\{ \frac{\uparrow^U \{x\}}{x \in U} \right\}$. The set S consists only of principal filters. But $[S]$ contains (exercise!) some nonprincipal filters. □

By the way:

Proposition 282. $\left\{ \frac{\bigsqcup^{\mathfrak{A}} X}{X \in \mathscr{P}S} \right\}$ is closed under binary meets, if S is a strong partition of an element of a complete lattice.

PROOF. Let $R = \left\{ \frac{\bigsqcup^{\mathfrak{A}} X}{X \in \mathscr{P}S} \right\}$. Then for every $X, Y \in \mathscr{P}S$

$$\begin{aligned}
\bigsqcup^{\mathfrak{A}} X \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y &= \\
\bigsqcup^{\mathfrak{A}} ((X \cap Y) \cup (X \setminus Y)) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y &= \\
\left(\bigsqcup^{\mathfrak{A}} (X \cap Y) \sqcup^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} (X \setminus Y) \right) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y &= \\
\left(\bigsqcup^{\mathfrak{A}} (X \cap Y) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y \right) \sqcup^{\mathfrak{A}} \left(\bigsqcup^{\mathfrak{A}} (X \setminus Y) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y \right) &= \\
\left(\bigsqcup^{\mathfrak{A}} (X \cap Y) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y \right) \sqcup^{\mathfrak{A}} \bot^{\mathfrak{A}} &= \\
\bigsqcup^{\mathfrak{A}} (X \cap Y) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y.
\end{aligned}$$

Applying the formula $\bigsqcup^{\mathfrak{A}} X \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y = \bigsqcup^{\mathfrak{A}} (X \cap Y) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y$ twice we get

$$\begin{aligned}
\bigsqcup^{\mathfrak{A}} X \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y &= \\
\bigsqcup^{\mathfrak{A}} (X \cap Y) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} (Y \cap (X \cap Y)) &= \\
\bigsqcup^{\mathfrak{A}} (X \cap Y) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} (X \cap Y) &= \\
\bigsqcup^{\mathfrak{A}} (X \cap Y).
\end{aligned}$$

But for any $A, B \in R$ there exist $X, Y \in \mathscr{P}S$ such that $A = \bigsqcup^{\mathfrak{A}} X$, $B = \bigsqcup^{\mathfrak{A}} Y$. So $A \sqcap^{\mathfrak{A}} B = \bigsqcup^{\mathfrak{A}} X \sqcap \bigsqcup^{\mathfrak{A}} Y = \bigsqcup^{\mathfrak{A}} (X \cap Y) \in R$. □

3.6. A proposition about binary relations

PROPOSITION 283. Let f, g, h be binary relations. Then $g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}$.

PROOF.

$$\begin{aligned}
g \circ f \not\asymp h &\Leftrightarrow \\
\exists a, c : a \; ((g \circ f) \cap h) \; c &\Leftrightarrow \\
\exists a, c : (a \; (g \circ f) \; c \wedge a \; h \; c) &\Leftrightarrow \\
\exists a, b, c : (a \; f \; b \wedge b \; g \; c \wedge a \; h \; c) &\Leftrightarrow \\
\exists b, c : (b \; g \; c \wedge b \; (h \circ f^{-1}) \; c) &\Leftrightarrow \\
\exists b, c : b \; (g \cap (h \circ f^{-1})) \; c &\Leftrightarrow \\
g \not\asymp h \circ f^{-1}.
\end{aligned}$$

□

3.7. Infinite associativity and ordinated product

3.7.1. Introduction. We will consider some function f which takes an arbitrary ordinal number of arguments. That is f can be taken for arbitrary (small, if to be precise) ordinal number of arguments. More formally: Let $x = x_{i \in n}$ be a

family indexed by an ordinal n. Then $f(x)$ can be taken. The same function f can take different number of arguments. (See below for the exact definition.)

Some of such functions f are associative in the sense defined below. If a function is associative in the below defined sense, then the binary operation induced by this function is associative in the usual meaning of the word "associativity" as defined in basic algebra.

I also introduce and research an important example of infinitely associative function, which I call *ordinated product*.

Note that my searching about infinite associativity and ordinals in Internet has provided no useful results. As such there is a reason to assume that my research of generalized associativity in terms of ordinals is novel.

3.7.2. Used notation. We identify natural numbers with finite Von Neumann's ordinals (further just *ordinals* or *ordinal numbers*).

For simplicity we will deal with small sets (members of a Grothendieck universe). We will denote the Grothendieck universe (aka *universal set*) as $\mho$.

I will denote a tuple of n elements like $[\![a_0, \dots, a_{n-1}]\!]$. By definition

$$[\![a_0, \dots, a_{n-1}]\!] = \{(0, a_0), \dots, (n-1, a_{n-1})\}.$$

Note that an ordered pair (a, b) is not the same as the tuple $[\![a, b]\!]$ of two elements. (However, we will use them interchangeably.)

Definition 284. An *anchored relation* is a tuple $[\![n, r]\!]$ where n is an index set and r is an n-ary relation.

For an anchored relation $\operatorname{arity}[\![n, r]\!] = n$. The graph[1] of $[\![n, r]\!]$ is defined as follows: $\operatorname{GR}[\![n, r]\!] = r$.

Definition 285. $\operatorname{Pr}_i f$ is a function defined by the formula

$$\operatorname*{Pr}_i f = \left\{ \frac{x_i}{x \in f} \right\}$$

for every small n-ary relation f where n is an ordinal number and $i \in n$. Particularly for every n-ary relation f and $i \in n$ where $n \in \mathbb{N}$

$$\operatorname*{Pr}_i f = \left\{ \frac{x_i}{[\![x_0, \dots, x_{n-1}]\!] \in f} \right\}.$$

Recall that Cartesian product is defined as follows:

$$\prod a = \left\{ \frac{z \in (\bigcup \operatorname{im} a)^{\operatorname{dom} a}}{\forall i \in \operatorname{dom} a : z(i) \in a_i} \right\}.$$

Obvious 286. If a is a small function, then $\prod a = \left\{ \frac{z \in \mho^{\operatorname{dom} a}}{\forall i \in \operatorname{dom} a : z(i) \in a_i} \right\}$.

3.7.2.1. *Currying and uncurrying.*

The customary definition. Let X, Y, Z be sets.

We will consider variables $x \in X$ and $y \in Y$.

Let a function $f \in Z^{X \times Y}$. Then $\operatorname{curry}(f) \in (Z^Y)^X$ is the function defined by the formula $(\operatorname{curry}(f)x)y = f(x, y)$.

Let now $f \in (Z^Y)^X$. Then $\operatorname{uncurry}(f) \in Z^{X \times Y}$ is the function defined by the formula $\operatorname{uncurry}(f)(x, y) = (fx)y$.

Obvious 287.

1°. $\operatorname{uncurry}(\operatorname{curry}(f)) = f$ for every $f \in Z^{X \times Y}$.
2°. $\operatorname{curry}(\operatorname{uncurry}(f)) = f$ for every $f \in (Z^Y)^X$.

[1] It is unrelated with graph theory.

Currying and uncurrying with a dependent variable. Let X, Z be sets and Y be a function with the domain X. (Vaguely saying, Y is a variable dependent on X.)

The disjoint union $\coprod Y = \bigcup_{i\in\operatorname{dom} Y}(\{i\}\times Y_i) = \left\{\frac{(i,x)}{i\in\operatorname{dom} Y, x\in Y_i}\right\}$.

We will consider variables $x\in X$ and $y\in Y_x$.

Let a function $f\in Z^{\coprod_{i\in X} Y_i}$ (or equivalently $f\in Z^{\coprod Y}$). Then $\operatorname{curry}(f)\in \prod_{i\in X} Z^{Y_i}$ is the function defined by the formula $(\operatorname{curry}(f)x)y = f(x,y)$.

Let now $f\in\prod_{i\in X} Z^{Y_i}$. Then $\operatorname{uncurry}(f)\in Z^{\coprod_{i\in X} Y_i}$ is the function defined by the formula $\operatorname{uncurry}(f)(x,y) = (fx)y$.

Obvious 288.

1°. $\operatorname{uncurry}(\operatorname{curry}(f)) = f$ for every $f\in Z^{\coprod_{i\in X} Y_i}$.
2°. $\operatorname{curry}(\operatorname{uncurry}(f)) = f$ for every $f\in\prod_{i\in X} Z^{Y_i}$.

3.7.2.2. *Functions with ordinal numbers of arguments.* Let Ord be the set of small ordinal numbers.

If X and Y are sets and n is an ordinal number, the set of functions taking n arguments on the set X and returning a value in Y is Y^{X^n}.

The set of all small functions taking ordinal numbers of arguments is $Y^{\bigcup_{n\in\operatorname{Ord}} X^n}$.

I will denote $\operatorname{OrdVar}(X) = \mho^{\bigcup_{n\in\operatorname{Ord}} X^n}$ and call it *ordinal variadic*. ("Var" in this notation is taken from the word *variadic* in the collocation *variadic function* used in computer science.)

3.7.3. On sums of ordinals. Let a be an ordinal-indexed family of ordinals.

Proposition 289. $\coprod a$ with lexicographic order is a well-ordered set.

Proof. Let S be non-empty subset of $\coprod a$.

Take $i_0 = \min \operatorname{Pr}_0 S$ and $x_0 = \min\left\{\frac{\operatorname{Pr}_1 y}{y\in S, y(0)=i_0}\right\}$ (these exist by properties of ordinals). Then (i_0, x_0) is the least element of S. □

Definition 290. $\sum a$ is the unique ordinal order-isomorphic to $\coprod a$.

Exercise 291. Prove that for finite ordinals it is just a sum of natural numbers.

This ordinal exists and is unique because our set is well-ordered.

Remark 292. An infinite sum of ordinals is not customary defined.

The *structured sum* $\bigoplus a$ of a is an order isomorphism from lexicographically ordered set $\coprod a$ into $\sum a$.

There exists (for a given a) exactly one structured sum, by properties of well-ordered sets.

Obvious 293. $\sum a = \operatorname{im}\bigoplus a$.

Theorem 294. $(\bigoplus a)(n,x) = \sum_{i\in n} a_i + x$.

Proof. We need to prove that it is an order isomorphism. Let's prove it is an injection that is $m>n \Rightarrow \sum_{i\in m} a_i + x > \sum_{i\in n} a_i + x$ and $y>x\Rightarrow \sum_{i\in n} a_i + y > \sum_{i\in n} a_i + x$.

Really, if $m>n$ then $\sum_{i\in m} a_i + x \ge \sum_{i\in n+1} a_i + x > \sum_{i\in n} a_i + x$. The second formula is true by properties of ordinals.

Let's prove that it is a surjection. Let $r\in\sum a$. There exist $n\in\operatorname{dom} a$ and $x\in a_n$ such that $r = (\bigoplus a)(n,x)$. Thus $r = (\bigoplus a)(n,0) + x = \sum_{i\in n} a_i + x$ because $(\bigoplus a)(n,0) = \sum_{i\in n} a_i$ since $(n,0)$ has $\sum_{i\in n} a_i$ predecessors. □

3.7.4. Ordinated product.

3.7.4.1. *Introduction. Ordinated product* defined below is a variation of Cartesian product, but is associative unlike Cartesian product. However, ordinated product unlike Cartesian product is defined not for arbitrary sets, but only for relations having ordinal numbers of arguments.

Let F indexed by an ordinal number be a small family of anchored relations.

3.7.4.2. *Concatenation.*

Definition 295. Let z be an indexed by an ordinal number family of functions each taking an ordinal number of arguments. The *concatenation* of z is

$$\operatorname{concat} z = \operatorname{uncurry}(z) \circ \left(\bigoplus(\operatorname{dom} \circ z)\right)^{-1}.$$

Exercise 296. Prove, that if z is a finite family of finitary tuples, it is concatenation of $\operatorname{dom} z$ tuples in the usual sense (as it is commonly used in computer science).

Proposition 297. If $z \in \prod(\operatorname{GR} \circ F)$ then $\operatorname{concat} z = \operatorname{uncurry}(z) \circ (\bigoplus(\operatorname{arity} \circ F))^{-1}$.

Proof. If $z \in \prod(\operatorname{GR} \circ F)$ then $\operatorname{dom} z(i) = \operatorname{dom}(\operatorname{GR} \circ F)_i = \operatorname{arity} F_i$ for every $i \in \operatorname{dom} F$. Thus $\operatorname{dom} \circ z = \operatorname{arity} \circ F$. □

Proposition 298. $\operatorname{dom} \operatorname{concat} z = \sum_{i \in \operatorname{dom} z} \operatorname{dom} z_i$.

Proof. Because $\operatorname{dom}(\bigoplus(\operatorname{dom} \circ z))^{-1} = \sum_{i \in \operatorname{dom} f}(\operatorname{dom} \circ z)$, it is enough to prove that

$$\operatorname{dom} \operatorname{uncurry}(z) = \operatorname{dom} \bigoplus(\operatorname{dom} \circ z).$$

Really,

$$\sum_{i \in \operatorname{dom} f} (\operatorname{dom} \circ z) = \left\{ \frac{(i, x)}{i \in \operatorname{dom}(\operatorname{dom} \circ z), x \in \operatorname{dom} z_i} \right\} = \left\{ \frac{(i, x)}{i \in \operatorname{dom} z, x \in \operatorname{dom} z_i} \right\} = \coprod z$$

and $\operatorname{dom} \operatorname{uncurry}(z) = \coprod_{i \in X} z_i = \coprod z$. □

3.7.4.3. *Finite example.* If F is a finite family (indexed by a natural number $\operatorname{dom} F$) of anchored finitary relations, then by definition

$$\operatorname{GR} \prod^{(\mathrm{ord})} = \left\{ \frac{[\![a_{0,0}, \ldots, a_{0,\operatorname{arity} F_0 - 1}, \ldots, a_{\operatorname{dom} F - 1, 0}, \ldots, a_{\operatorname{dom} F - 1, \operatorname{arity} F_{\operatorname{dom} F - 1} - 1}]\!]}{\begin{array}{c}[\![a_{0,0}, \ldots, a_{0,\operatorname{arity} F_0 - 1}]\!] \in \operatorname{GR} F_0 \wedge \ldots \wedge \\ [\![a_{\operatorname{dom} F - 1, \operatorname{arity} F_{\operatorname{dom} F - 1} - 1}]\!] \in \operatorname{GR} F_{\operatorname{dom} F - 1}\end{array}} \right\}$$

and

$$\operatorname{arity} \prod^{(\mathrm{ord})} F = \operatorname{arity} F_0 + \ldots + \operatorname{arity} F_{\operatorname{dom} F - 1}.$$

The above formula can be shortened to

$$\operatorname{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{\operatorname{concat} z}{z \in \prod(\operatorname{GR} \circ F)} \right\}.$$

3.7.4.4. *The definition.*

DEFINITION 299. The anchored relation (which I call *ordinated product*) $\prod^{(\mathrm{ord})} F$ is defined by the formulas:

$$\operatorname{arity} \overset{(\mathrm{ord})}{\prod} F = \sum(\operatorname{arity} \circ f);$$

$$\operatorname{GR} \overset{(\mathrm{ord})}{\prod} F = \left\{ \frac{\operatorname{concat} z}{z \in \prod(\operatorname{GR} \circ F)} \right\}.$$

PROPOSITION 300. $\prod^{(\mathrm{ord})} F$ is a properly defined anchored relation.

PROOF. $\operatorname{dom} \operatorname{concat} z = \sum_{i \in \operatorname{dom} F} \operatorname{dom} z_i = \sum_{i \in \operatorname{dom} F} \operatorname{arity} f_i = \sum(\operatorname{arity} \circ F)$. □

3.7.4.5. *Definition with composition for every multiplier.*

$$q(F)_i \overset{\mathrm{def}}{=} \left(\operatorname{curry}\left(\bigoplus(\operatorname{arity} \circ F)\right)\right) i.$$

PROPOSITION 301. $\prod^{(\mathrm{ord})} F = \left\{ \frac{L \in \mho^{\sum(\operatorname{arity} \circ F)}}{\forall i \in \operatorname{dom} F : L \circ q(F)_i \in \operatorname{GR} F_i} \right\}$.

PROOF. $\operatorname{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{\operatorname{concat} z}{z \in \prod(\operatorname{GR} \circ F)} \right\}$;

$\operatorname{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{\operatorname{uncurry}(z) \circ \left(\bigoplus(\operatorname{arity} \circ f)\right)^{-1}}{z \in \prod_{i \in \operatorname{dom} F} \mho^{\operatorname{arity} F_i}, \forall i \in \operatorname{dom} F : z(i) \in \operatorname{GR} F_i} \right\}$.

Let $L = \operatorname{uncurry}(z)$. Then $z = \operatorname{curry}(L)$.

$\operatorname{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{L \circ \left(\bigoplus(\operatorname{arity} \circ f)\right)^{-1}}{\operatorname{curry}(L) \in \prod_{i \in \operatorname{dom} F} \mho^{\operatorname{arity} F_i}, \forall i \in \operatorname{dom} F : \operatorname{curry}(L) i \in \operatorname{GR} F_i} \right\}$;

$\operatorname{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{L \circ \left(\bigoplus(\operatorname{arity} \circ f)\right)^{-1}}{L \in \mho^{\coprod_{i \in \operatorname{dom} F} \operatorname{arity} F_i}, \forall i \in \operatorname{dom} F : \operatorname{curry}(L) i \in \operatorname{GR} F_i} \right\}$;

$\operatorname{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{L \in \mho^{\sum(\operatorname{arity} \circ f)}}{\forall i \in \operatorname{dom} F : \operatorname{curry}\left(L \circ \bigoplus(\operatorname{arity} \circ F)\right) i \in \operatorname{GR} F_i} \right\}$;

$(\operatorname{curry}(L \circ \bigoplus(\operatorname{arity} \circ F)) i) x = L((\operatorname{curry}(\bigoplus(\operatorname{arity} \circ F)) i) x) = L(q(F)_i x) = (L \circ q(F)_i) x$;

$\operatorname{curry}(L \circ \bigoplus(\operatorname{arity} \circ F)) i = L \circ q(F)_i$;

$\prod^{(\mathrm{ord})} F = \left\{ \frac{L \in \mho^{\sum(\operatorname{arity} \circ F)}}{\forall i \in \operatorname{dom} F : L \circ q(F)_i \in \operatorname{GR} F_i} \right\}$. □

COROLLARY 302. $\prod^{(\mathrm{ord})} F = \left\{ \frac{L \in \left(\bigcup \operatorname{im}(\operatorname{GR} \circ F)\right)^{\sum(\operatorname{arity} \circ F)}}{\forall i \in \operatorname{dom} F : L \circ q(F)_i \in \operatorname{GR} F_i} \right\}$.

COROLLARY 303. $\prod^{(\mathrm{ord})} F$ is small if F is small.

3.7.4.6. *Definition with shifting arguments.* Let $F_i' = \left\{ \frac{L \circ \operatorname{Pr}_1 |_{\{i\} \times \operatorname{arity} F_i}}{L \in \operatorname{GR} F_i} \right\}$.

PROPOSITION 304. $F_i' = \left\{ \frac{L \circ \operatorname{Pr}_1 |_{\{i\} \times \mho}}{L \in \operatorname{GR} F_i} \right\}$.

PROOF. If $L \in \operatorname{GR} F_i$ then $\operatorname{dom} L = \operatorname{arity} F_i$. Thus

$$L \circ \operatorname*{Pr}_1 |_{\{i\} \times \operatorname{arity} F_i} = L \circ \operatorname*{Pr}_1 |_{\{i\} \times \operatorname{dom} L} = L \circ \operatorname*{Pr}_1 |_{\{i\} \times \mho}.$$

□

PROPOSITION 305. F_i' is an $(\{i\} \times \operatorname{arity} F_i)$-ary relation.

Proof. We need to prove that $\operatorname{dom}\bigl(L \circ \operatorname{Pr}_1 |_{\{i\}\times\operatorname{arity} F_i}\bigr) = \{i\} \times \operatorname{arity} F_i$ for $L \in \operatorname{GR} F_i$, but that's obvious. □

Obvious 306. $\coprod(\operatorname{arity} \circ F) = \bigcup_{i\in\operatorname{dom} F}(\{i\} \times \operatorname{arity} F_i) = \bigcup_{i\in\operatorname{dom} F} \operatorname{dom} F'_i$.

Lemma 307. $P \in \prod_{i\in\operatorname{dom} F} F'_i \Leftrightarrow \operatorname{curry}(\bigcup \operatorname{im} P) \in \prod(\operatorname{GR} \circ F)$ for a $(\operatorname{dom} F)$-indexed family P where $P_i \in \mho^{\{i\}\times\operatorname{arity} F_i}$ for every $i \in \operatorname{dom} F$, that is for $P \in \coprod_{i\in\operatorname{dom} F} \mho^{\{i\}\times\operatorname{arity} F_i}$.

Proof. For every $P \in \coprod_{i\in\operatorname{dom} F} \mho^{\{i\}\times\operatorname{arity} F_i}$ we have:

$$
\begin{aligned}
P \in \prod_{i\in\operatorname{dom} F} F'_i &\Leftrightarrow \\
P \in \left\{\frac{z \in \mho^{\operatorname{dom} F}}{\forall i \in \operatorname{dom} F : z(i) \in F'_i}\right\} &\Leftrightarrow \\
P \in \mho^{\operatorname{dom} F} \wedge \forall i \in \operatorname{dom} F : P(i) \in F'_i &\Leftrightarrow \\
P \in \mho^{\operatorname{dom} F} \wedge \forall i \in \operatorname{dom} F \exists L \in \operatorname{GR} F_i : P_i = L \circ (\operatorname{Pr}_1 |_{\{i\}\times\mho}) &\Leftrightarrow \\
P \in \mho^{\operatorname{dom} F} \wedge \forall i \in \operatorname{dom} F \exists L \in \operatorname{GR} F_i : (P_i \in \mho^{\{i\}\times\operatorname{arity} F_i} \wedge \forall x \in \operatorname{arity} F_i : P_i(i,x) = Lx) &\Leftrightarrow \\
P \in \mho^{\operatorname{dom} F} \wedge \forall i \in \operatorname{dom} F \exists L \in \operatorname{GR} F_i : (P_i \in \mho^{\{i\}\times\operatorname{arity} F_i} \wedge \operatorname{curry}(P_i)i = L) &\Leftrightarrow \\
P \in \mho^{\operatorname{dom} F} \wedge \forall i \in \operatorname{dom} F : (P_i \in \mho^{\{i\}\times\operatorname{arity} F_i} \wedge \operatorname{curry}(P_i)i \in \operatorname{GR} F_i) &\Leftrightarrow \\
\forall i \in \operatorname{dom} F \exists Q_i \in (\mho^{\operatorname{arity} F_i})^{\{i\}} : (P_i = \operatorname{uncurry}(Q_i) \wedge (Q_i)i \in \mho^{\operatorname{arity} F_i} \wedge Q_i i \in \operatorname{GR} F_i) &\Leftrightarrow \\
\forall i \in \operatorname{dom} F \exists Q_i \in (\mho^{\operatorname{arity} F_i})^{\{i\}} : \left(P_i = \operatorname{uncurry}(Q_i) \wedge \left(\bigcup_{i\in\operatorname{dom} F} Q_i\right) i \in \operatorname{GR} F_i\right) &\Leftrightarrow \\
\forall i \in \operatorname{dom} F \exists Q_i \in (\mho^{\operatorname{arity} F_i})^{\{i\}} : \left(P_i = \operatorname{uncurry}(Q_i) \wedge \bigcup_{i\in\operatorname{dom} F} Q_i \in \prod(\operatorname{GR} \circ F)\right) &\Leftrightarrow \\
\forall i \in \operatorname{dom} F : \bigcup_{i\in\operatorname{dom} F} \operatorname{curry}(P_i) \in \prod(\operatorname{GR} \circ F) &\Leftrightarrow \\
\operatorname{curry}\left(\bigcup_{i\in\operatorname{dom} F} P_i\right) \in \prod(\operatorname{GR} \circ F) &\Leftrightarrow \\
\operatorname{curry}\left(\bigcup \operatorname{im} P\right) \in \prod(\operatorname{GR} \circ F).&
\end{aligned}
$$

□

Lemma 308. $\left\{\frac{\operatorname{curry}(f)\circ\bigoplus(\operatorname{arity} \circ F)}{f\in\operatorname{GR}\prod^{(\operatorname{ord})} F}\right\} = \prod(\operatorname{GR} \circ F)$.

Proof. First $\operatorname{GR}\prod^{(\operatorname{ord})} F = \left\{\frac{\operatorname{uncurry}(z)\circ\left(\bigoplus(\operatorname{dom} \circ z)\right)^{-1}}{z\in\prod(\operatorname{GR} \circ F)}\right\}$, that is $\left\{\frac{f}{f\in\operatorname{GR}\prod^{(\operatorname{ord})} F}\right\} = \left\{\frac{\operatorname{uncurry}(z)\circ\left(\bigoplus(\operatorname{arity} \circ F)\right)^{-1}}{z\in\prod(\operatorname{GR} \circ F)}\right\}$.
Since $\bigoplus(\operatorname{arity} \circ F)$ is a bijection, we have
$\left\{\frac{f\circ\bigoplus(\operatorname{arity} \circ F)}{f\in\operatorname{GR}\prod^{(\operatorname{ord})} F}\right\} = \left\{\frac{\operatorname{uncurry}(z)}{z\in\prod(\operatorname{GR} \circ F)}\right\}$ what is equivalent to
$\left\{\frac{\operatorname{curry}(f)\circ\bigoplus(\operatorname{arity} \circ F)}{f\in\operatorname{GR}\prod^{(\operatorname{ord})} F}\right\} = \left\{\frac{z}{z\in\prod(\operatorname{GR} \circ F)}\right\}$ that is $\left\{\frac{\operatorname{curry}(f)\circ\bigoplus(\operatorname{arity} \circ F)}{f\in\operatorname{GR}\prod^{(\operatorname{ord})} F}\right\} = \prod(\operatorname{GR} \circ F)$. □

Lemma 309. $\left\{\frac{\bigcup \operatorname{im} P}{P\in\coprod_{i\in\operatorname{dom} F} \mho^{\{i\}\times\operatorname{arity} F_i} \wedge \operatorname{curry}(\bigcup \operatorname{im} P)\in\prod(\operatorname{GR} \circ F)}\right\} = \left\{\frac{L\in\mho^{\coprod_{i\in\operatorname{dom} F} \operatorname{arity} F_i}}{\operatorname{curry}(L)\in\prod(\operatorname{GR} \circ F)}\right\}$.

Proof. Let $L' \in \left\{\frac{L\in\mho^{\coprod_{i\in\operatorname{dom}F}\operatorname{arity}F_i}}{\operatorname{curry}(L)\in\prod(\operatorname{GR}\circ F)}\right\}$. Then $L' \in \mho^{\coprod_{i\in\operatorname{dom}F}\operatorname{arity}F_i}$ and $\operatorname{curry}(L') \in \prod(\operatorname{GR}\circ F)$.

Let $P = \lambda i \in \operatorname{dom}F : L'|_{\{i\}\times\operatorname{arity}F_i}$. Then $P \in \coprod_{i\in\operatorname{dom}F}\mho^{\{i\}\times\operatorname{arity}F_i}$ and $\bigcup\operatorname{im}P = L'$. So $L' \in \left\{\frac{\bigcup\operatorname{im}P}{P\in\coprod_{i\in\operatorname{dom}F}\mho^{\{i\}\times\operatorname{arity}F_i}\wedge\operatorname{curry}(\bigcup\operatorname{im}P)\in\prod(\operatorname{GR}\circ F)}\right\}$.

Let now $L' \in \left\{\frac{\bigcup\operatorname{im}P}{P\in\coprod_{i\in\operatorname{dom}F}\mho^{\{i\}\times\operatorname{arity}F_i}\wedge\operatorname{curry}(\bigcup\operatorname{im}P)\in\prod(\operatorname{GR}\circ F)}\right\}$. Then there exists $P \in \coprod_{i\in\operatorname{dom}F}\mho^{\{i\}\times\operatorname{arity}F_i}$ such that $L' = \bigcup\operatorname{im}P$ and $\operatorname{curry}(L') \in \prod(\operatorname{GR}\circ F)$. Evidently $L' \in \mho^{\coprod_{i\in\operatorname{dom}F}\operatorname{arity}F_i}$. So $L' \in \left\{\frac{L\in\mho^{\coprod_{i\in\operatorname{dom}F}\operatorname{arity}F_i}}{\operatorname{curry}(L)\in\prod(\operatorname{GR}\circ F)}\right\}$. □

Lemma 310. $\left\{\frac{f\circ\bigoplus(\operatorname{arity}\circ F)}{f\in\operatorname{GR}\prod^{(\operatorname{ord})}F}\right\} = \left\{\frac{\bigcup\operatorname{im}P}{P\in\prod_{i\in\operatorname{dom}F}F'_i}\right\}$.

Proof.

$$L \in \left\{\frac{\bigcup\operatorname{im}P}{P\in\prod_{i\in\operatorname{dom}F}F'_i}\right\} \Leftrightarrow$$

$$L \in \left\{\frac{\bigcup\operatorname{im}P}{P\in\coprod_{i\in\operatorname{dom}F}\mho^{\{i\}\times\operatorname{arity}F_i}\wedge\operatorname{curry}(\bigcup\operatorname{im}P)\in\prod(\operatorname{GR}\circ F)}\right\} \Leftrightarrow$$

$$L \in \mho^{\coprod_{i\in\operatorname{dom}F}\operatorname{arity}F_i}\wedge\operatorname{curry}(L)\in\prod(\operatorname{GR}\circ F) \Leftrightarrow$$

$$L \in \mho^{\coprod_{i\in\operatorname{dom}F}\operatorname{arity}F_i}\wedge\operatorname{curry}(L)\in\left\{\frac{\operatorname{curry}(f)\circ\bigoplus(\operatorname{arity}\circ F)}{f\in\operatorname{GR}\prod^{(\operatorname{ord})}F}\right\} \Leftrightarrow$$

(because $\bigoplus(\operatorname{arity}\circ F)$ is a bijection)

$$\operatorname{curry}(L)\circ\left(\bigoplus(\operatorname{arity}\circ F)\right)^{-1}\in\left\{\frac{\operatorname{curry}(f)}{f\in\operatorname{GR}\prod^{(\operatorname{ord})}F}\right\} \Leftrightarrow$$

$$L\circ\left(\bigoplus(\operatorname{arity}\circ F)\right)^{-1}\in\left\{\frac{f}{f\in\operatorname{GR}\prod^{(\operatorname{ord})}F}\right\} \Leftrightarrow$$

(because $\bigoplus(\operatorname{arity}\circ F)$ is a bijection)

$$L \in \left\{\frac{f\circ\bigoplus(\operatorname{arity}\circ F)}{f\in\operatorname{GR}\prod^{(\operatorname{ord})}F}\right\}.$$

□

Theorem 311. $\operatorname{GR}\prod^{(\operatorname{ord})}F = \left\{\frac{(\bigcup\operatorname{im}P)\circ\left(\bigoplus(\operatorname{arity}\circ F)\right)^{-1}}{P\in\prod_{i\in\operatorname{dom}F}F'_i}\right\}$.

Proof. From the lemma, because $\bigoplus(\operatorname{arity}\circ F)$ is a bijection. □

Theorem 312. $\operatorname{GR}\prod^{(\operatorname{ord})}F = \left\{\frac{\bigcup_{i\in\operatorname{dom}F}\left(P_i\circ\left(\bigoplus(\operatorname{arity}\circ f)\right)^{-1}\right)}{P\in\prod_{i\in\operatorname{dom}F}F'_i}\right\}$.

Proof. From the previous theorem. □

Theorem 313. $\operatorname{GR}\prod^{(\operatorname{ord})}F = \left\{\frac{\bigcup\operatorname{im}P}{P\in\prod_{i\in\operatorname{dom}F}\left\{\frac{f\circ\left(\bigoplus(\operatorname{arity}\circ f)\right)^{-1}}{f\in F'_i}\right\}}\right\}$.

Proof. From the previous. □

REMARK 314. Note that the above formulas contain both $\bigcup_{i\in\operatorname{dom} F}\operatorname{dom} F'_i$ and $\bigcup_{i\in\operatorname{dom} F} F'_i$. These forms are similar but different.

3.7.4.7. *Associativity of ordinated product.* Let f be an ordinal variadic function.

Let S be an ordinal indexed family of functions of ordinal indexed families of functions each taking an ordinal number of arguments in a set X.

I call f *infinite associative* when

1°. $f(f\circ S) = f(\operatorname{concat} S)$ for every S;
2°. $f([\![x]\!]) = x$ for $x\in X$.

Infinite associativity implies associativity.

PROPOSITION 315. Let f be an infinitely associative function taking an ordinal number of arguments in a set X. Define $x\star y = f[\![x,y]\!]$ for $x,y\in X$. Then the binary operation $\star$ is associative.

PROOF. Let $x,y,z\in X$. Then $(x\star y)\star z = f[\![f[\![x,y]\!],z]\!] = f(f[\![x,y]\!],f[\![z]\!]) = f[\![x,y,z]\!]$. Similarly $x\star(y\star z) = f[\![x,y,z]\!]$. So $(x\star y)\star z = x\star(y\star z)$. □

Concatenation is associative. First we will prove some lemmas.

Let a and b be functions on a poset. Let $a\sim b$ iff there exist an order isomorphism f such that $a = b\circ f$. Evidently $\sim$ is an equivalence relation.

OBVIOUS 316. $\operatorname{concat} a = \operatorname{concat} b \Leftrightarrow \operatorname{uncurry}(a)\sim\operatorname{uncurry}(b)$ for every ordinal indexed families a and b of functions taking an ordinal number of arguments.

Thank to the above, we can reduce properties of concat to properties of uncurry.

LEMMA 317. $a\sim b\Rightarrow \operatorname{uncurry} a\sim\operatorname{uncurry} b$ for every ordinal indexed families a and b of functions taking an ordinal number of arguments.

PROOF. There exists an order isomorphism f such that $a = b\circ f$.

$$\operatorname{uncurry}(a)(x,y) = (ax)y = (bfx)y = \operatorname{uncurry}(b)(fx,y) = \operatorname{uncurry}(b)g(x,y)$$

where $g(x,y) = (fx,y)$.

g is an order isomorphism because $g(x_0,y_0)\ge g(x_1,y_1)\Leftrightarrow (x_0,y_0)\ge(x_1,y_1)$. (Injectivity and surjectivity are obvious.) □

LEMMA 318. Let $a_i\sim b_i$ for every i. Then $\operatorname{uncurry} a\sim\operatorname{uncurry} b$ for every ordinal indexed families a and b of ordinal indexed families of functions taking an ordinal number of arguments.

PROOF. Let $a_i = b_i\circ f_i$ where f_i is an order isomorphism for every i.

$$\operatorname{uncurry}(a)(i,y) = a_iy = b_if_iy = \operatorname{uncurry}(b)(i,f_iy) = \operatorname{uncurry}(b)g(i,y) = (\operatorname{uncurry}(b)\circ g)(i,y)$$

where $g(i,y) = (i,f_iy)$.

g is an order isomorphism because $g(i,y_0)\ge g(i,y_1)\Leftrightarrow f_iy_0\ge f_iy_1\Leftrightarrow y_0\ge y_1$ and $i_0>i_1\Rightarrow g(i,y_0)>g(i,y_1)$. (Injectivity and surjectivity are obvious.) □

Let now S be an ordinal indexed family of ordinal indexed families of functions taking an ordinal number of arguments.

LEMMA 319. $\operatorname{uncurry}(\operatorname{uncurry}\circ S)\sim\operatorname{uncurry}(\operatorname{uncurry} S)$.

PROOF. $\operatorname{uncurry} \circ S = \lambda i \in S : \operatorname{uncurry}(S_i)$;
$(\operatorname{uncurry}(\operatorname{uncurry} \circ S))((i,x),y) = (\operatorname{uncurry} S_i)(x,y) = (S_i x)y$;
$(\operatorname{uncurry}(\operatorname{uncurry} S))((i,x),y) = ((\operatorname{uncurry} S)(i,x))y = (S_i x)y$.

Thus $(\operatorname{uncurry}(\operatorname{uncurry} \circ S))((i,x),y) = (\operatorname{uncurry}(\operatorname{uncurry} S))((i,x),y)$ and thus evidently $\operatorname{uncurry}(\operatorname{uncurry} \circ S) \sim \operatorname{uncurry}(\operatorname{uncurry} S)$. □

THEOREM 320. concat is an infinitely associative function.

PROOF. $\operatorname{concat}([\![x]\!]) = x$ for a function x taking an ordinal number of argument is obvious. It is remained to prove

$$\operatorname{concat}(\operatorname{concat} \circ S) = \operatorname{concat}(\operatorname{concat} S);$$

We have, using the lemmas,

$$\begin{gathered}\operatorname{concat}(\operatorname{concat} \circ S) \sim \\ \operatorname{uncurry}(\operatorname{concat} \circ S) \sim \\ \text{(by lemma 318)} \\ \operatorname{uncurry}(\operatorname{uncurry} \circ S) \sim \\ \operatorname{uncurry}(\operatorname{uncurry} S) \sim \\ \operatorname{uncurry}(\operatorname{concat} S) \sim \\ \operatorname{concat}(\operatorname{concat} S).\end{gathered}$$

Consequently $\operatorname{concat}(\operatorname{concat} \circ S) = \operatorname{concat}(\operatorname{concat} S)$. □

COROLLARY 321. Ordinated product is an infinitely associative function.

3.8. Galois surjections

DEFINITION 322. *Galois surjection* is the special case of Galois connection such that $f^* \circ f_*$ is identity.

PROPOSITION 323. For Galois surjection $\mathfrak{A} \to \mathfrak{B}$ such that $\mathfrak{A}$ is a join-semilattice we have (for every $y \in \mathfrak{B}$)

$$f_* y = \max\left\{ \frac{x \in \mathfrak{A}}{f^* x = y} \right\}.$$

PROOF. We need to prove (theorem 131)

$$\max\left\{ \frac{x \in \mathfrak{A}}{f^* x = y} \right\} = \max\left\{ \frac{x \in \mathfrak{A}}{f^* x \sqsubseteq y} \right\}.$$

To prove it, it's enough to show that for each $f^* x \sqsubseteq y$ there exists an $x' \sqsupseteq x$ such that $f^* x' = y$.

Really, $y = f^* f_* y$. It's enough to prove $f^*(x \sqcup f_* y) = y$.

Indeed (because lower adjoints preserve joins), $f^*(x \sqcup f_* y) = f^* x \sqcup f^* f_* y = f^* x \sqcup y = y$. □

3.9. Some properties of frames

This section is based on a TODD TRIMBLE's proof. A shorter but less elementary proof (also by TODD TRIMBLE) is available at
http://ncatlab.org/toddtrimble/published/topogeny

I will abbreviate *join-semilattice with least element* as JSWLE.

OBVIOUS 324. JSWLEs are the same as finitely join-closed posets (with nullary joins included).

Definition 325. It is said that a function f from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ *preserves finite joins*, when for every finite set $S \in \mathscr{P}\mathfrak{A}$ such that $\bigsqcup^{\mathfrak{A}} S$ exists we have $\bigsqcup^{\mathfrak{B}} \langle f \rangle^* S = f \bigsqcup^{\mathfrak{A}} S$.

Obvious 326. A function between JSWLEs preserves finite joins iff it preserves binary joins ($f(x \sqcup y) = fx \sqcup fy$) and nullary joins ($f(\bot^{\mathfrak{A}}) = \bot^{\mathfrak{B}}$).

Definition 327. A *fixed point* of a function F is such x that $F(x) = x$. We will denote $\operatorname{Fix}(F)$ the set of all fixed points of a function F.

Definition 328. Let $\mathfrak{A}$ be a JSWLE. A *co-nucleus* is a function $F : \mathfrak{A} \to \mathfrak{A}$ such that for every $p, q \in \mathfrak{A}$ we have:

1°. $F(p) \sqsubseteq p$;
2°. $F(F(p)) = F(p)$;
3°. $F(p \sqcup q) = F(p) \sqcup F(q)$.

Proposition 329. Every co-nucleus is a monotone function.

Proof. It follows from $F(p \sqcup q) = F(p) \sqcup F(q)$. □

Lemma 330. $\bigsqcup^{\operatorname{Fix}(F)} S = \bigsqcup S$ for every $S \in \mathscr{P} \operatorname{Fix}(F)$ for every co-nucleus F on a complete lattice.

Proof. Obviously $\bigsqcup S \sqsupseteq x$ for every $x \in S$.

Suppose $z \sqsupseteq x$ for every $x \in S$ for a $z \in \operatorname{Fix}(F)$. Then $z \sqsupseteq \bigsqcup S$.

$F(\bigsqcup S) \sqsupseteq F(x)$ for every $x \in S$. Thus $F(\bigsqcup S) \sqsupseteq \bigsqcup_{x \in S} F(x) = \bigsqcup S$. But $F(\bigsqcup S) \sqsubseteq \bigsqcup S$. Thus $F(\bigsqcup S) = \bigsqcup S$ that is $\bigsqcup S \in \operatorname{Fix}(F)$.

So $\bigsqcup^{\operatorname{Fix}(F)} S = \bigsqcup S$ by the definition of join. □

Corollary 331. $\bigsqcup^{\operatorname{Fix}(F)} S$ is defined for every $S \in \mathscr{P} \operatorname{Fix}(F)$.

Lemma 332. $\bigsqcap^{\operatorname{Fix}(F)} S = F(\bigsqcap S)$ for every $S \in \mathscr{P} \operatorname{Fix}(F)$ for every co-nucleus F on a complete lattice.

Proof. Obviously $F(\bigsqcap S) \sqsubseteq x$ for every $x \in S$.

Suppose $z \sqsubseteq x$ for every $x \in S$ for a $z \in \operatorname{Fix}(F)$. Then $z \sqsubseteq \bigsqcap S$ and thus $z \sqsubseteq F(\bigsqcap S)$.

So $\bigsqcap^{\operatorname{Fix}(F)} S = F(\bigsqcap S)$ by the definition of meet. □

Corollary 333. $\bigsqcap^{\operatorname{Fix}(F)} S$ is defined for every $S \in \mathscr{P} \operatorname{Fix}(F)$.

Obvious 334. $\operatorname{Fix}(F)$ with induced order is a complete lattice.

Lemma 335. If F is a co-nucleus on a co-frame $\mathfrak{A}$, then the poset $\operatorname{Fix}(F)$ of fixed points of F, with order inherited from $\mathfrak{A}$, is also a co-frame.

PROOF. Let $b \in \operatorname{Fix}(F)$, $S \in \mathscr{P}\operatorname{Fix}(F)$. Then

$$\begin{aligned} b \sqcup^{\operatorname{Fix}(F)} \bigsqcap^{\operatorname{Fix}(F)} S &= \\ b \sqcup^{\operatorname{Fix}(F)} F\Big(\bigsqcap S\Big) &= \\ F(b) \sqcup F\Big(\bigsqcap S\Big) &= \\ F\Big(b \sqcup \bigsqcap S\Big) &= \\ F\Big(\bigsqcap \langle b\sqcup\rangle^* S\Big) &= \\ \bigsqcap^{\operatorname{Fix}(F)} \langle b\sqcup\rangle^* S &= \\ \bigsqcap^{\operatorname{Fix}(F)} \langle b\sqcup^{\operatorname{Fix}(F)}\rangle^* S. \end{aligned}$$

□

DEFINITION 336. Denote $\operatorname{Up}(\mathfrak{A})$ the set of upper sets on $\mathfrak{A}$ ordered *reverse* to set theoretic inclusion.

DEFINITION 337. Denote $\uparrow a = \left\{ \frac{x \in \mathfrak{A}}{x \sqsupseteq a} \right\} \in \operatorname{Up}(\mathfrak{A})$.

LEMMA 338. The set $\operatorname{Up}(\mathfrak{A})$ is closed under arbitrary meets and joins.

PROOF. Let $S \in \mathscr{P}\operatorname{Up}(\mathfrak{A})$.

Let $X \in \bigcup S$ and $Y \sqsupseteq X$ for an $Y \in \mathfrak{A}$. Then there is $P \in S$ such that $X \in P$ and thus $Y \in P$ and so $Y \in \bigcup S$. So $\bigcup S \in \operatorname{Up}(\mathfrak{A})$.

Let now $X \in \bigcap S$ and $Y \sqsupseteq X$ for an $Y \in \mathfrak{A}$. Then $\forall T \in S : X \in T$ and so $\forall T \in S : Y \in T$, thus $Y \in \bigcap S$. So $\bigcap S \in \operatorname{Up}(\mathfrak{A})$. □

THEOREM 339. A poset $\mathfrak{A}$ is a complete lattice iff there is a antitone map $s : \operatorname{Up}(\mathfrak{A}) \to \mathfrak{A}$ such that

1°. $s(\uparrow p) = p$ for every $p \in \mathfrak{A}$;
2°. $D \subseteq \uparrow s(D)$ for every $D \in \operatorname{Up}(\mathfrak{A})$.

Moreover, in this case $s(D) = \bigsqcap D$ for every $D \in \operatorname{Up}(\mathfrak{A})$.

PROOF.

$\Rightarrow$. Take $s(D) = \bigsqcap D$.

$\Leftarrow$. $\forall x \in D : x \sqsupseteq s(D)$ from the second formula.

Let $\forall x \in D : y \sqsubseteq x$. Then $x \in \uparrow y$, $D \subseteq \uparrow y$; because s is an antitone map, thus follows $s(D) \sqsupseteq s(\uparrow y) = y$. So $\forall x \in D : y \sqsubseteq s(D)$.

That s is the meet follows from the definition of meets.

It remains to prove that $\mathfrak{A}$ is a complete lattice.

Take any subset S of $\mathfrak{A}$. Let D be the smallest upper set containing S. (It exists because $\operatorname{Up}(\mathfrak{A})$ is closed under arbitrary joins.) This is

$$D = \left\{ \frac{x \in \mathfrak{A}}{\exists s \in S : x \sqsupseteq s} \right\}.$$

Any lower bound of D is clearly a lower bound of S since $D \supseteq S$. Conversely any lower bound of S is a lower bound of D. Thus S and D have the same set of lower bounds, hence have the same greatest lower bound.

□

PROPOSITION 340. For any poset $\mathfrak{A}$ the following are mutually reverse order isomorphisms between upper sets F (ordered reverse to set-theoretic inclusion) on $\mathfrak{A}$ and order homomorphisms $\varphi : \mathfrak{A}^{\mathrm{op}} \to 2$ (here 2 is the partially ordered set of two elements: 0 and 1 where $0 \sqsubseteq 1$), defined by the formulas

1°. $\varphi(a) = \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{cases}$ for every $a \in \mathfrak{A}$;

2°. $F = \varphi^{-1}(1)$.

PROOF. Let $X \in \varphi^{-1}(1)$ and $Y \sqsupseteq X$. Then $\varphi(X) = 1$ and thus $\varphi(Y) = 1$. Thus $\varphi^{-1}(1)$ is a upper set.

It is easy to show that φ defined by the formula 1° is an order homomorphism $\mathfrak{A}^{\mathrm{op}} \to 2$ whenever F is a upper set.

Finally we need to prove that they are mutually inverse. Really: Let φ be defined by the formula 1°. Then take $F' = \varphi^{-1}(1)$ and define $\varphi'(a)$ by the formula 1°. We have

$$\varphi'(a) = \begin{cases} 1 & \text{if } a \in \varphi^{-1}(1) \\ 0 & \text{if } a \notin \varphi^{-1}(1) \end{cases} = \begin{cases} 1 & \text{if } \varphi(a) = 1 \\ 0 & \text{if } \varphi(a) \neq 1 \end{cases} = \varphi(a).$$

Let now F be defined by the formula 2°. Then take $\varphi'(a) = \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{cases}$ as defined by the formula 1° and define $F' = \varphi'^{-1}(1)$. Then

$$F' = \varphi'^{-1}(1) = F.$$

□

LEMMA 341. For a complete lattice $\mathfrak{A}$, the map $\bigsqcap : \mathrm{Up}(\mathfrak{A}) \to \mathfrak{A}$ preserves arbitrary meets.

PROOF. Let $S \in \mathscr{P}\,\mathrm{Up}(\mathfrak{A})$. We have $\bigsqcap S \in \mathrm{Up}(\mathfrak{A})$.
$\bigsqcap\bigsqcap S = \bigsqcap\bigsqcap_{X \in S} X = \bigsqcap_{X \in S}\bigsqcap X$ is what we needed to prove. □

LEMMA 342. A complete lattice $\mathfrak{A}$ is a co-frame iff $\bigsqcap : \mathrm{Up}(\mathfrak{A}) \to \mathfrak{A}$ preserves finite joins.

PROOF.

⇒. Let $\mathfrak{A}$ be a co-frame. Let $D, D' \in \mathrm{Up}(\mathfrak{A})$. Obviously $\bigsqcap(D \sqcup D') \sqsupseteq \bigsqcap D$ and $\bigsqcap(D \sqcup D') \sqsupseteq \bigsqcap D'$, so $\bigsqcap(D \sqcup D') \sqsupseteq \bigsqcap D \sqcup \bigsqcap D'$.

Also

$$\bigsqcap D \sqcup \bigsqcap D' = \bigsqcup D \sqcup \bigsqcup D' = (\text{because } \mathfrak{A} \text{ is a co-frame}) = \bigsqcup\left\{\frac{d \sqcup d'}{d \in D, d' \in D'}\right\}.$$

Obviously $d \sqcup d' \in D \cap D'$, thus $\bigsqcap D \sqcup \bigsqcap D' \subseteq \bigsqcup(D \cap D') = \bigsqcap(D \cap D')$ that is $\bigsqcap D \sqcup \bigsqcap D' \sqsupseteq \bigsqcap(D \cap D')$. So $\bigsqcap(D \sqcup D') = \bigsqcap D \sqcup \bigsqcap D'$ that is $\bigsqcap : \mathrm{Up}(A) \to A$ preserves binary joins.

It preserves nullary joins since $\bigsqcap^{\mathrm{Up}(\mathfrak{A})} \bot^{\mathrm{Up}(\mathfrak{A})} = \bigsqcap^{\mathrm{Up}(\mathfrak{A})} \mathfrak{A} = \bot^{\mathfrak{A}}$.

$\Leftarrow$. Suppose $\bigsqcap : \mathrm{Up}(\mathfrak{A}) \to \mathfrak{A}$ preserves finite joins. Let $b \in \mathfrak{A}$, $S \in \mathscr{P}\mathfrak{A}$. Let D be the smallest upper set containing S (so $D = \bigcup \langle\uparrow\rangle^* S$). Then

$$\begin{aligned}
b \sqcup \bigsqcap S &= \\
\bigsqcap \uparrow b \sqcup \bigcup \bigsqcap \langle\uparrow\rangle^* S &= \\
\bigsqcap \uparrow b \sqcup \bigsqcap \bigcup \langle\uparrow\rangle^* S &= (\text{since } \bigsqcap \text{ preserves finite joins}) \\
\bigsqcap \Big(\uparrow b \sqcup \bigcup \langle\uparrow\rangle^* S\Big) &= \\
\bigcup \Big(\uparrow b \cap \bigcup \langle\uparrow\rangle^* S\Big) &= \\
\bigsqcap \bigcup_{a \in S} (\uparrow b \cap \uparrow a) &= \\
\bigsqcap \bigcup_{a \in S} \uparrow (b \sqcup a) &= (\text{since } \bigsqcap \text{ preserves all meets}) \\
\bigcup_{a \in S} \bigsqcap \uparrow (b \sqcup a) &= \\
\bigcup_{a \in S} (b \sqcup a) &= \\
\bigsqcap_{a \in S} (b \sqcup a).
\end{aligned}$$

$\square$

Corollary 343. If $\mathfrak{A}$ is a co-frame, then the composition $F = \uparrow \circ \bigsqcap : \mathrm{Up}(\mathfrak{A}) \to \mathrm{Up}(\mathfrak{A})$ is a co-nucleus. The embedding $\uparrow : \mathfrak{A} \to \mathrm{Up}(\mathfrak{A})$ is an isomorphism of $\mathfrak{A}$ onto the co-frame $\mathrm{Fix}(F)$.

Proof. $D \sqsupseteq F(D)$ follows from theorem 339.

We have $F(F(D)) = F(D)$ for all $D \in \mathrm{Up}(\mathfrak{A})$ since $F(F(D)) = \uparrow \bigsqcap \uparrow \bigsqcap D =$ (because $\bigsqcap \uparrow s = s$ for any s) $= \uparrow \bigsqcap D = F(D)$.

And since both $\bigsqcap : \mathrm{Up}(\mathfrak{A}) \to \mathfrak{A}$ and $\uparrow$ preserve finite joins, F preserves finite joins. Thus F is a co-nucleus.

Finally, we have $a \sqsupseteq a'$ if and only if $\uparrow a \subseteq \uparrow a'$, so that $\uparrow : \mathfrak{A} \to \mathrm{Up}(\mathfrak{A})$ maps $\mathfrak{A}$ isomorphically onto its image $\langle\uparrow\rangle^* \mathfrak{A}$. This image is $\mathrm{Fix}(F)$ because if D is any fixed point (i.e. if $D = \uparrow \bigsqcap D$), then D clearly belongs to $\langle\uparrow\rangle^* \mathfrak{A}$; and conversely $\uparrow a$ is always a fixed point of $F = \uparrow \circ \bigsqcap$ since $F(\uparrow a) = \uparrow \bigsqcap \uparrow a = \uparrow a$. $\square$

Definition 344. If $\mathfrak{A}$, $\mathfrak{B}$ are two JSWLEs, then $\mathrm{Join}(\mathfrak{A}, \mathfrak{B})$ is the (ordered pointwise) set of finite joins preserving maps $\mathfrak{A} \to \mathfrak{B}$.

Obvious 345. $\mathrm{Join}(\mathfrak{A}, \mathfrak{B})$ is a JSWLE, where $f \sqcup g$ is given by the formula $(f \sqcup g)(p) = f(p) \sqcup g(p)$, $\bot^{\mathrm{Join}(\mathfrak{A},\mathfrak{B})}$ is given by the formula $\bot^{\mathrm{Join}(\mathfrak{A},\mathfrak{B})}(p) = \bot^{\mathfrak{B}}$.

Definition 346. Let $h : Q \to R$ be a finite joins preserving map. Then by definition $\mathrm{Join}(P, h) : \mathrm{Join}(P, Q) \to \mathrm{Join}(P, R)$ takes $f \in \mathrm{Join}(P, Q)$ into the composition $h \circ f \in \mathrm{Join}(P, R)$.

Lemma 347. Above defined $\mathrm{Join}(P, h)$ is a finite joins preserving map.

Proof.

$$\begin{aligned}
(h \circ (f \sqcup f'))x = h(f \sqcup f')x = h(fx \sqcup f'x) = \\
hfx \sqcup hf'x = (h \circ f)x \sqcup (h \circ f')x = ((h \circ f) \sqcup (h \circ f'))x.
\end{aligned}$$

Thus $h \circ (f \sqcup f') = (h \circ f) \sqcup (h \circ f')$.

$(h \circ \bot^{\mathrm{Join}(P,Q)})x = h\bot^{\mathrm{Join}(P,Q)}x = h\bot^Q = \bot^R$. □

PROPOSITION 348. If $h, h' : Q \to R$ are finite join preserving maps and $h \sqsupseteq h'$, then $\mathrm{Join}(P, h) \sqsupseteq \mathrm{Join}(P, h')$.

PROOF. $\mathrm{Join}(P, h)(f)(x) = (h \circ f)(x) = hfx \sqsupseteq h'fx = (h' \circ f)(x) = \mathrm{Join}(P, h')(f)(x)$. □

LEMMA 349. If $g : Q \to R$ and $h : R \to S$ are finite joins preserving, then the composition $\mathrm{Join}(P, h) \circ \mathrm{Join}(P, g)$ is equal to $\mathrm{Join}(P, h \circ g)$. Also $\mathrm{Join}(P, \mathrm{id}_Q)$ for identity map id_Q on Q is the identity map $\mathrm{id}_{\mathrm{Join}(P,Q)}$ on $\mathrm{Join}(P, Q)$.

PROOF. $\mathrm{Join}(P, h)\,\mathrm{Join}(P, g)f = \mathrm{Join}(P, h)(g \circ f) = h \circ g \circ f = \mathrm{Join}(P, h \circ g)f$.

$\mathrm{Join}(P, \mathrm{id}_Q)f = \mathrm{id}_Q \circ f = f$. □

COROLLARY 350. If Q is a JSWLE and $F : Q \to Q$ is a co-nucleus, then for any JSWLE P we have that

$$\mathrm{Join}(P, F) : \mathrm{Join}(P, Q) \to \mathrm{Join}(P, Q)$$

is also a co-nucleus.

PROOF. From $\mathrm{id}_Q \sqsupseteq F$ (co-nucleus axiom 1°) we have $\mathrm{Join}(P, \mathrm{id}_Q) \sqsupseteq \mathrm{Join}(P, F)$ and since by the last lemma the left side is the identity on $\mathrm{Join}(P, Q)$, we see that $\mathrm{Join}(P, F)$ also satisfies co-nucleus axiom 1°.

$\mathrm{Join}(P, F) \circ \mathrm{Join}(P, F) = \mathrm{Join}(P, F \circ F)$ by the same lemma and thus $\mathrm{Join}(P, F) \circ \mathrm{Join}(P, F) = \mathrm{Join}(P, F)$ by the second co-nucleus axiom for F, showing that $\mathrm{Join}(P, F)$ satisfies the second co-nucleus axiom.

By an other lemma, we have that $\mathrm{Join}(P, F)$ preserves binary joins, given that F preserves binary joins, which is the third co-nucleus axiom. □

LEMMA 351. $\mathrm{Fix}(\mathrm{Join}(P, F)) = \mathrm{Join}(P, \mathrm{Fix}(F))$ for every JSWLEs P, Q and a join preserving function $F : Q \to Q$.

PROOF. $a \in \mathrm{Fix}(\mathrm{Join}(P, F)) \Leftrightarrow a \in F^P \wedge F \circ a = a \Leftrightarrow a \in F^P \wedge \forall x \in P : F(a(x)) = a(x)$.

$a \in \mathrm{Join}(P, \mathrm{Fix}(F)) \Leftrightarrow a \in \mathrm{Fix}(F)^P \Leftrightarrow a \in F^P \wedge \forall x \in P : F(a(x)) = a(x)$.

Thus $\mathrm{Fix}(\mathrm{Join}(P, F)) = \mathrm{Join}(P, \mathrm{Fix}(F))$. That the order of the left and right sides of the equality agrees is obvious. □

DEFINITION 352. $\mathbf{Pos}(\mathfrak{A}, \mathfrak{B})$ is the pointwise ordered poset of monotone maps from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.

LEMMA 353. If Q, R are JSWLEs and P is a poset, then $\mathbf{Pos}(P, R)$ is a JSWLE and $\mathbf{Pos}(P, \mathrm{Join}(Q, R))$ is isomorphic to $\mathrm{Join}(Q, \mathbf{Pos}(P, R))$. If R is a co-frame, then also $\mathbf{Pos}(P, R)$ is a co-frame.

PROOF. Let $f, g \in \mathbf{Pos}(P, R)$. Then $\lambda x \in P : (fx \sqcup gx)$ is obviously monotone and then it is evident that $f \sqcup^{\mathbf{Pos}(P,R)} g = \lambda x \in P : (fx \sqcup gx)$. $\lambda x \in P : \bot^R$ is also obviously monotone and it is evident that $\bot^{\mathbf{Pos}(P,R)} = \lambda x \in P : \bot^R$.

Obviously both $\mathbf{Pos}(P, \mathrm{Join}(Q, R))$ and $\mathrm{Join}(Q, \mathbf{Pos}(P, R))$ are sets of order preserving maps.

Let f be a monotone map.

$f \in \mathbf{Pos}(P, \mathrm{Join}(Q, R))$ iff $f \in \mathrm{Join}(Q, R)^P$ iff $f \in \left\{\frac{g \in R^Q}{g \text{ preserves finite joins}}\right\}^P$ iff $f \in (R^Q)^P$ and every $g = f(x)$ (for $x \in P$) preserving finite joins. This is bijectively equivalent $(f \mapsto f')$ to $f' \in (R^P)^Q$ preserving finite joins.

$f' \in \operatorname{Join}(Q, \mathbf{Pos}(P,R))$ iff f' preserves finite joins and $f' \in \mathbf{Pos}(P,R)^Q$ iff f' preserves finite joins and $f' \in \left\{ \frac{g \in (R^P)^Q}{g(x) \text{ is monotone}} \right\}$ iff f' preserves finite joins and $f' \in (R^P)^Q$.

So we have proved that $f \mapsto f'$ is a bijection between $\mathbf{Pos}(P, \operatorname{Join}(Q,R))$ and $\operatorname{Join}(Q, \mathbf{Pos}(P,R))$. That it preserves order is obvious.

It remains to prove that if R is a co-frame, then also $\mathbf{Pos}(P,R)$ is a co-frame.

First, we need to prove that $\mathbf{Pos}(P,R)$ is a complete lattice. But it is easy to prove that for every set $S \in \mathscr{P}\mathbf{Pos}(P,R)$ we have $\lambda x \in P : \bigsqcup_{f \in S} f(x)$ and $\lambda x \in P : \bigsqcap_{f \in S} f(x)$ are monotone and thus are the joins and meets on $\mathbf{Pos}(P,R)$.

Next we need to prove that

$$b \sqcup^{\mathbf{Pos}(P,R)} \bigsqcap^{\mathbf{Pos}(P,R)} S = \bigsqcap^{\mathbf{Pos}(P,R)} \left\langle b \sqcup^{\mathbf{Pos}(P,R)} \right\rangle^* S.$$

Really (for every $x \in P$),

$$\left(b \sqcup^{\mathbf{Pos}(P,R)} \bigsqcap^{\mathbf{Pos}(P,R)} S \right) x = b(x) \sqcup \left(\bigsqcap^{\mathbf{Pos}(P,R)} S \right) x =$$

$$b(x) \sqcup \bigsqcap_{f \in S} f(x) = \bigsqcap_{f \in S} (b(x) \sqcup f(x)) = \bigsqcap_{f \in S} \left(b \sqcup^{\mathbf{Pos}(P,R)} f \right) x =$$

$$\left(\bigsqcap_{f \in S}^{\mathbf{Pos}(P,R)} \left(b \sqcup^{\mathbf{Pos}(P,R)} f \right) \right) x.$$

Thus

$$b \sqcup^{\mathbf{Pos}(P,R)} \bigsqcap^{\mathbf{Pos}(P,R)} S =$$

$$\bigsqcap_{f \in S}^{\mathbf{Pos}(P,R)} \left(b \sqcup^{\mathbf{Pos}(P,R)} f \right) = \bigsqcap^{\mathbf{Pos}(P,R)} \left\langle b \sqcup^{\mathbf{Pos}(P,R)} \right\rangle^* S.$$

□

Definition 354. $P \cong Q$ means that posets P and Q are isomorphic.

CHAPTER 4

Typed sets and category Rel

4.1. Relational structures

DEFINITION 355. A *relational structure* is a pair consisting of a set and a tuple of relations on this set.

A poset $(\mathfrak{A}, \sqsubseteq)$ can be considered as a relational structure: $(\mathfrak{A}, [\![\sqsubseteq]\!])$.

A set can X be considered as a relational structure with zero relations: $(X, [\![]\!])$.

This book is not about relational structures. So I will not introduce more examples.

Think about relational structures as a common place for sets or posets, as far as they are considered in this book.

We will denote $x \in (\mathfrak{A}, R)$ iff $x \in \mathfrak{A}$ for a relational structure $(\mathfrak{A}, R)$.

4.2. Typed elements and typed sets

We sometimes want to differentiate between the same element of two different sets. For example, we may want to consider different the natural number 3 and the rational number 3. In order to describe this in a formal way we consider elements of sets together with sets themselves. For example, we can consider the pairs $(\mathbb{N}, 3)$ and $(\mathbb{Q}, 3)$.

DEFINITION 356. A *typed element* is a pair $(\mathfrak{A}, a)$ where $\mathfrak{A}$ is a relational structure and $a \in \mathfrak{A}$.

I denote $\operatorname{type}(\mathfrak{A}, a) = \mathfrak{A}$ and $\operatorname{GR}(\mathfrak{A}, a) = a$.

DEFINITION 357. I will denote typed element $(\mathfrak{A}, a)$ as $@^{\mathfrak{A}}a$ or just $@a$ when $\mathfrak{A}$ is clear from context.

DEFINITION 358. A *typed set* is a typed element equal to $(\mathscr{P}U, A)$ where U is a set and A is its subset.

REMARK 359. *Typed sets* is an awkward formalization of type theory sets in ZFC (U is meant to express the *type* of the set). This book could be better written using type theory instead of ZFC, but I want my book to be understandable for everyone knowing ZFC. $(\mathscr{P}U, A)$ should be understood as a set A of type U. For an example, consider $(\mathscr{P}\mathbb{R}, [0; 10])$; it is the closed interval $[0; 10]$ whose elements are considered as real numbers.

DEFINITION 360. $\mathfrak{T}\mathfrak{A} = \left\{ \frac{(\mathfrak{A}, a)}{a \in \mathfrak{A}} \right\} = \{\mathfrak{A}\} \times \mathfrak{A}$ for every relational structure $\mathfrak{A}$.

REMARK 361. $\mathfrak{T}\mathfrak{A}$ is the set of typed elements of $\mathfrak{A}$.

DEFINITION 362. If $\mathfrak{A}$ is a poset, we introduce order on its typed elements isomorphic to the order of the original poset: $(\mathfrak{A}, a) \sqsubseteq (\mathfrak{A}, b) \Leftrightarrow a \sqsubseteq b$.

DEFINITION 363. I denote $\operatorname{GR}(\mathfrak{A}, a) = a$ for a typed element $(\mathfrak{A}, a)$.

DEFINITION 364. I will denote *typed subsets* of a typed poset $(\mathscr{P}U, A)$ as $\mathscr{P}(\mathscr{P}U, A) = \left\{ \frac{(\mathscr{P}U, X)}{X \in \mathscr{P}A} \right\} = \{\mathscr{P}U\} \times \mathscr{P}A$.

Obvious 365. $\mathscr{P}(\mathscr{P}U, A)$ is also a set of typed sets.

Definition 366. I will denote $\mathscr{T}U = \mathfrak{T}\mathscr{P}U$.

Remark 367. This means that $\mathscr{T}U$ is the set of typed subsets of a set U.

Obvious 368. $\mathscr{T}U = \left\{ \frac{(\mathscr{P}U, X)}{X \in \mathscr{P}U} \right\} = \{\mathscr{P}U\} \times \mathscr{P}U = \mathscr{P}(\mathscr{P}U, U)$.

Obvious 369. $\mathscr{T}U$ is a complete atomistic boolean lattice. Particularly:

1°. $\bot^{\mathscr{T}U} = (\mathscr{P}U, \emptyset)$;
2°. $\top^{\mathscr{T}U} = (\mathscr{P}U, U)$;
3°. $(\mathscr{P}U, A) \sqcup (\mathscr{P}U, B) = (\mathscr{P}U, A \cup B)$;
4°. $(\mathscr{P}U, A) \sqcap (\mathscr{P}U, B) = (\mathscr{P}U, A \cap B)$;
5°. $\bigsqcup_{A \in S}(\mathscr{P}U, A) = (\mathscr{P}U, \bigcup_{A \in S} A)$;
6°. $\bigsqcap_{A \in S}(\mathscr{P}U, A) = \left(\mathscr{P}U, \begin{cases} \bigcap_{A \in S} A & \text{if } A \neq \emptyset \\ U & \text{if } A = \emptyset \end{cases} \right)$;
7°. $\overline{(\mathscr{P}U, A)} = (\mathscr{P}U, U \setminus A)$;
8°. atomic elements are $(\mathscr{P}U, \{x\})$ where $x \in U$.

Typed sets are "better" than regular sets as (for example) for a set U and a typed set X the following are defined by regular order theory:

- $\operatorname{atoms} X$;
- $\overline{X}$;
- $\bigsqcap^{\mathscr{T}U} \emptyset$.

For regular ("non-typed") sets these are not defined (except of $\operatorname{atoms} X$ which however needs a special definition instead of using the standard order-theory definition of atoms).

Typed sets are convenient to be used together with filters on sets (see below), because both typed sets and filters have a set $\mathscr{P}U$ as their type.

Another advantage of typed sets is that their binary product (as defined below) is a **Rel**-morphism. This is especially convenient because below defined products of filters are also morphisms of related categories.

Well, typed sets are also quite awkward, but the proper way of doing modern mathematics is *type theory* not ZFC, what is however outside of the topic of this book.

4.3. Category Rel

I remind that **Rel** is the category of (small) binary relations between sets, and **Set** is its subcategory where only monovalued entirely defined morphisms (functions) are considered.

Definition 370. Order on $\mathbf{Rel}(A, B)$ is defined by the formula $f \sqsubseteq g \Leftrightarrow \operatorname{GR} f \subseteq \operatorname{GR} g$.

Obvious 371. This order is isomorphic to the natural order of subsets of the set $A \times B$.

Definition 372. $X\ [f]^*\ Y \Leftrightarrow \operatorname{GR} X\ [\operatorname{GR} f]^*\ \operatorname{GR} Y$ and $\langle f \rangle^* X = (\operatorname{Dst} f, \langle \operatorname{GR} f \rangle^* \operatorname{GR} X)$ for a **Rel**-morphism f and typed sets $X \in \mathscr{T} \operatorname{Src} f$, $Y \in \mathscr{T} \operatorname{Dst} f$.

Definition 373. For category **Rel** there is defined reverse morphism: $(A, B, F)^{-1} = (B, A, F^{-1})$.

Obvious 374. $(f^{-1})^{-1} = f$ for every **Rel**-morphism f.

OBVIOUS 375. $\left[f^{-1}\right]^*=[f]^{*-1}$ for every **Rel**-morphism f.

OBVIOUS 376. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for every composable **Rel**-morphisms f and g.

PROPOSITION 377. $\langle g \circ f \rangle^* = \langle g \rangle^* \circ \langle f \rangle^*$ for every composable **Rel**-morphisms f and g.

PROOF. Exercise. □

PROPOSITION 378. The above definitions of monovalued morphisms of **Rel** and of injective morphisms of **Set** coincide with how mathematicians usually define monovalued functions (that is morphisms of **Set**) and injective functions.

PROOF. Let f be a **Rel**-morphism $A \to B$.
The following are equivalent:

- f is a monovalued relation;
- $\forall x \in A, y_0, y_1 \in B : (x \ f \ y_0 \wedge x \ f \ y_1 \Rightarrow y_0 = y_1)$;
- $\forall x \in A, y_0, y_1 \in B : (y_0 \neq y_1 \Rightarrow \neg(x \ f \ y_0) \vee \neg(x \ f \ y_1))$;
- $\forall y_0, y_1 \in B \forall x \in A : (y_0 \neq y_1 \Rightarrow \neg(x \ f \ y_0) \vee \neg(x \ f \ y_1))$;
- $\forall y_0, y_1 \in B : (y_0 \neq y_1 \Rightarrow \forall x \in A : (\neg(x \ f \ y_0) \vee \neg(x \ f \ y_1)))$;
- $\forall y_0, y_1 \in B : (\exists x \in A : (x \ f \ y_0 \wedge x \ f \ y_1) \Rightarrow y_0 = y_1)$;
- $\forall y_0, y_1 \in B : y_0 \ (f \circ f^{-1}) \ y_1 \Rightarrow y_0 = y_1$;
- $f \circ f^{-1} \sqsubseteq 1_B$.

Let now f be a **Set**-morphism $A \to B$.
The following are equivalent:

- f is an injective function;
- $\forall y \in B, a, b \in A : (a \ f \ y \wedge b \ f \ y \Rightarrow a = b)$;
- $\forall y \in B, a, b \in A : (a \neq b \Rightarrow \neg(a \ f \ y) \vee \neg(b \ f \ y))$;
- $\forall y \in B : (a \neq b \Rightarrow \forall a, b \in A : (\neg(a \ f \ y) \vee \neg(b \ f \ y)))$;
- $\forall y \in B : (\exists a, b \in A : (a \ f \ y \wedge b \ f \ y) \Rightarrow a = b)$;
- $f^{-1} \circ f \sqsubseteq 1_A$.

□

PROPOSITION 379. For a binary relation f we have:

1°. $\langle f \rangle^* \bigcup S = \bigcup \left\langle \langle f \rangle^* \right\rangle^* S$ for a set of sets S;
2°. $\bigcup S \ [f]^* \ Y \Leftrightarrow \exists X \in S : X \ [f]^* \ Y$ for a set of sets S;
3°. $X \ [f]^* \bigcup T \Leftrightarrow \exists Y \in T : X \ [f]^* \ Y$ for a set of sets T;
4°. $\bigcup S \ [f]^* \bigcup T \Leftrightarrow \exists X \in S, Y \in T : X \ [f]^* \ Y$ for sets of sets S and T;
5°. $X \ [f]^* \ Y \Leftrightarrow \exists \alpha \in X, \beta \in Y : \{\alpha\} \ [f]^* \ \{\beta\}$ for sets X and Y;
6°. $\langle f \rangle^* X = \bigcup \left\langle \langle f \rangle^* \right\rangle^* \operatorname{atoms} X$ for a set X (where $\operatorname{atoms} X = \left\{ \frac{\{x\}}{x \in X} \right\}$).

PROOF.

1°.

$$y \in \langle f \rangle^* \bigcup S \Leftrightarrow \exists x \in \bigcup S : x \ f \ y \Leftrightarrow \exists P \in S, x \in P : x \ f \ y \Leftrightarrow$$
$$\exists P \in S : y \in \langle f \rangle^* P \Leftrightarrow \exists Q \in \left\langle \langle f \rangle^* \right\rangle^* S : y \in Q \Leftrightarrow y \in \bigcup \left\langle \langle f \rangle^* \right\rangle^* S.$$

2°.

$$\bigcup S \ [f]^* \ Y \Leftrightarrow \exists x \in \bigcup S, y \in Y : x \ f \ y \Leftrightarrow$$
$$\exists X \in S, x \in X, y \in Y : x \ f \ y \Leftrightarrow \exists X \in S : X \ [f]^* \ Y.$$

3°. By symmetry.

4°. From two previous formulas.
5°. $X\ [f]^*\ Y \Leftrightarrow \exists \alpha \in X, \beta \in Y : \alpha\ f\ \beta \Leftrightarrow \exists \alpha \in X, \beta \in Y : \{\alpha\}\ [f]^*\ \{\beta\}$.
6°. Obvious.

□

Corollary 380. For a **Rel**-morphism f we have:

1°. $\langle f\rangle^* \bigsqcup S = \bigsqcup \langle \langle f\rangle^* \rangle^* S$ for $S \in \mathscr{P}\mathscr{T}\operatorname{Src} f$;
2°. $\bigsqcup S\ [f]^*\ Y \Leftrightarrow \exists X \in S : X\ [f]^*\ Y$ for $S \in \mathscr{P}\mathscr{T}\operatorname{Src} f$;
3°. $X\ [f]^* \bigsqcup T \Leftrightarrow \exists Y \in T : X\ [f]^*\ Y$ for $T \in \mathscr{P}\mathscr{T}\operatorname{Dst} f$;
4°. $\bigsqcup S\ [f]^*\ \bigsqcup T \Leftrightarrow \exists X \in S, Y \in T : X\ [f]^*\ Y$ for $S \in \mathscr{P}\mathscr{T}\operatorname{Src} f$, $T \in \mathscr{P}\mathscr{T}\operatorname{Dst} f$;
5°. $X\ [f]^*\ Y \Leftrightarrow \exists x \in \operatorname{atoms} X, y \in \operatorname{atoms} Y : x\ [f]^*\ y$ for $X \in \mathscr{T}\operatorname{Src} f$, $Y \in \mathscr{T}\operatorname{Dst} f$;
6°. $\langle f\rangle^* X = \bigsqcup \langle \langle f\rangle^* \rangle^* \operatorname{atoms} X$ for $X \in \mathscr{T}\operatorname{Src} f$.

Corollary 381. A **Rel**-morphism f can be restored knowing either $\langle f\rangle^* x$ for atoms $x \in \mathscr{T}\operatorname{Src} f$ or $x\ [f]^*\ y$ for atoms $x \in \mathscr{T}\operatorname{Src} f$, $y \in \mathscr{T}\operatorname{Dst} f$.

Proposition 382. Let A, B be sets, R be a set of binary relations.

1°. $\langle \bigcup R\rangle^* X = \bigcup_{f\in R} \langle f\rangle^* X$ for every set X;
2°. $\langle \bigcap R\rangle^* \{\alpha\} = \bigcap_{f\in R} \langle f\rangle^* \{\alpha\}$ for every α, if R is nonempty;
3°. $X\ [\bigcup R]^*\ Y \Leftrightarrow \exists f \in R : X\ [f]^*\ Y$ for every sets X, Y;
4°. $\alpha\ (\bigcap R)\ \beta \Leftrightarrow \forall f \in R : \alpha\ f\ \beta$ for every α and β, if R is nonempty.

Proof.

1°.

$$y \in \left\langle \bigcup R \right\rangle^* X \Leftrightarrow \exists x \in X : x \left(\bigcup R\right) y \Leftrightarrow \exists x \in X, f \in R : x\ f\ y \Leftrightarrow \exists f \in R : y \in \langle f\rangle^* X \Leftrightarrow y \in \bigcup_{f\in R} \langle f\rangle^* X.$$

2°.

$$y \in \left\langle \bigcap R \right\rangle^* \{\alpha\} \Leftrightarrow \forall f \in R : \alpha\ f\ y \Leftrightarrow \forall f \in R : y \in \langle f\rangle^* \{\alpha\} \Leftrightarrow y \in \bigcap_{f\in R} \langle f\rangle^* \{\alpha\}.$$

3°.

$$X \left[\bigcup R\right]^* Y \Leftrightarrow \exists x \in X, y \in Y : x \left(\bigcup R\right) y \Leftrightarrow \exists x \in X, y \in Y, f \in R : x\ f\ y \Leftrightarrow \exists f \in R : X\ [f]^*\ Y.$$

4°. Obvious.

□

Corollary 383. Let A, B be sets, $R \in \mathscr{P}\mathbf{Rel}(A, B)$.

1°. $\langle \bigsqcup R\rangle^* X = \bigsqcup_{f\in R} \langle f\rangle^* X$ for $X \in \mathscr{T}A$;
2°. $\langle \bigsqcap R\rangle^* x = \bigsqcap_{f\in R} \langle f\rangle^* x$ for atomic $x \in \mathscr{T}A$;
3°. $X\ [\bigsqcup R]^*\ Y \Leftrightarrow \exists f \in R : X\ [f]^*\ Y$ for $X \in \mathscr{T}A$, $Y \in \mathscr{T}B$;
4°. $x\ [\bigsqcap R]^*\ y \Leftrightarrow \forall f \in R : x\ [f]^*\ y$ for every atomic $x \in \mathscr{T}A$, $y \in \mathscr{T}B$.

Proposition 384. $X\ [g \circ f]^*\ Z \Leftrightarrow \exists \beta : (X\ [f]^*\ \{\beta\} \wedge \{\beta\}\ [g]^*\ Z)$ for every binary relation f and sets X and Y.

PROOF.

$$X\ [g\circ f]^*\ Z \Leftrightarrow \exists x\in X, z\in Z : x\ (g\circ f)\ z \Leftrightarrow$$
$$\exists x\in X, z\in Z, \beta : (x\ f\ \beta \wedge \beta\ g\ z) \Leftrightarrow$$
$$\exists \beta : (\exists x\in X : x\ f\ \beta \wedge \exists y\in Y : \beta\ g\ z) \Leftrightarrow \exists \beta : (X\ [f]^*\ \{\beta\} \wedge \{\beta\}\ [g]^*\ Z).$$

□

COROLLARY 385. $X\ [g\circ f]^*\ Z \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathscr{T} B} : (X\ [f]^*\ y \wedge y\ [g]^*\ Z)$ for $f\in \mathbf{Rel}(A,B)$, $g\in\mathbf{Rel}(B,C)$ (for sets A, B, C).

PROPOSITION 386. $f\circ\bigcup G = \bigcup_{g\in G}(f\circ g)$ and $\bigcup G\circ f = \bigcup_{g\in G}(g\circ f)$ for every binary relation f and set G of binary relations.

PROOF. We will prove only $\bigcup G\circ f = \bigcup_{g\in G}(g\circ f)$ as the other formula follows from duality. Really

$$(x,z)\in \bigcup G\circ f \Leftrightarrow \exists y : ((x,y)\in f \wedge (y,z)\in \bigcup G) \Leftrightarrow$$
$$\exists y, g\in G : ((x,y)\in f \wedge (y,z)\in g) \Leftrightarrow \exists g\in G : (x,z)\in g\circ f \Leftrightarrow (x,z)\in \bigcup_{g\in G}(g\circ f).$$

□

COROLLARY 387. Every **Rel**-morphism is metacomplete and co-metacomplete.

PROPOSITION 388. The following are equivalent for a **Rel**-morphism f:

1°. f is monovalued.
2°. f is metamonovalued.
3°. f is weakly metamonovalued.
4°. $\langle f\rangle^* a$ is either atomic or least whenever $a\in \operatorname{atoms}^{\mathscr{T}\operatorname{Src} f}$.
5°. $\left\langle f^{-1}\right\rangle^*(I\sqcap J) = \left\langle f^{-1}\right\rangle^* I \sqcap \left\langle f^{-1}\right\rangle^* J$ for every $I, J\in \mathscr{T}\operatorname{Src} f$.
6°. $\left\langle f^{-1}\right\rangle^* \bigsqcap S = \bigsqcap_{Y\in S}\left\langle f^{-1}\right\rangle^* Y$ for every $S\in \mathscr{P}\mathscr{T}\operatorname{Src} f$.

PROOF.

2°⇒3°. Obvious.
1°⇒2°. Take $x\in \operatorname{atoms}^{\mathscr{T}\operatorname{Src} f}$; then $fx\in \operatorname{atoms}^{\mathscr{T}\operatorname{Dst} f}\cup\{\bot^{\mathscr{T}\operatorname{Dst} f}\}$ and thus

$$\left\langle \left(\bigsqcap G\right)\circ f\right\rangle^* x = \left\langle \bigsqcap G\right\rangle^* \langle f\rangle^* x = \bigsqcap_{g\in G}\langle g\rangle^*\langle f\rangle^* x =$$
$$\bigsqcap_{g\in G}\langle g\circ f\rangle^* x = \left\langle \bigsqcap_{g\in G}(g\circ f)\right\rangle^* x;$$

so $(\bigsqcap G)\circ f = \bigsqcap_{g\in G}(g\circ f)$.
3°⇒1°. Take $g=\{(a,y)\}$ and $h=\{(b,y)\}$ for arbitrary $a\neq b$ and arbitrary y. We have $g\cap h=\emptyset$; thus $(g\circ f)\cap(h\circ f) = (g\cap h)\circ f = \bot$ and thus impossible $x\ f\ a \wedge x\ f\ b$ as otherwise $(x,y)\in (g\circ f)\cap(h\circ f)$. Thus f is monovalued.

4°⇒6°. Let $a \in \operatorname{atoms}^{\mathscr{T}\operatorname{Src} f}$, $\langle f\rangle^* a = b$. Then because $b \in \operatorname{atoms}^{\mathscr{T}\operatorname{Dst} f} \cup \{\bot^{\mathscr{T}\operatorname{Dst} f}\}$

$$\bigsqcap S \sqcap b \neq \bot \Leftrightarrow \forall Y \in S : Y \sqcap b \neq \bot;$$

$$a\ [f]^*\ \bigsqcap S \Leftrightarrow \forall Y \in S : a\ [f]^*\ Y;$$

$$\bigsqcap S\ \left[f^{-1}\right]^*\ a \Leftrightarrow \forall Y \in S : Y\ \left[f^{-1}\right]^*\ a;$$

$$a \not\asymp \left\langle f^{-1}\right\rangle^* \bigsqcap S \Leftrightarrow \forall Y \in S : a \not\asymp \left\langle f^{-1}\right\rangle^* Y;$$

$$a \not\asymp \left\langle f^{-1}\right\rangle^* \bigsqcap S \Leftrightarrow a \not\asymp \bigsqcap_{Y \in S} \left\langle f^{-1}\right\rangle^* Y;$$

$$\left\langle f^{-1}\right\rangle^* \bigsqcap S = \bigsqcap_{X \in S} \left\langle f^{-1}\right\rangle^* X.$$

6°⇒5°. Obvious.

5°⇒1°. $\left\langle f^{-1}\right\rangle^* a \sqcap \left\langle f^{-1}\right\rangle^* b = \left\langle f^{-1}\right\rangle^* (a \sqcap b) = \left\langle f^{-1}\right\rangle^* \bot = \bot$ for every two distinct atoms $a = \{\alpha\}, b = \{\beta\} \in \mathscr{T}\operatorname{Dst} f$. From this

$$\alpha\ (f \circ f^{-1})\ \beta \Leftrightarrow \exists y \in \operatorname{Dst} f : (\alpha\ f^{-1}\ y \wedge y\ f\ \beta) \Leftrightarrow$$
$$\exists y \in \operatorname{Dst} f : (y \in \left\langle f^{-1}\right\rangle^* a \wedge y \in \left\langle f^{-1}\right\rangle^* b)$$

is impossible. Thus $f \circ f^{-1} \sqsubseteq 1^{\mathbf{Rel}}_{\operatorname{Dst} f}$.

¬4°⇒ ¬1°. Suppose $\langle f\rangle^* a \notin \operatorname{atoms}^{\mathscr{T}\operatorname{Dst} f} \cup \{\bot^{\mathscr{T}\operatorname{Dst} f}\}$ for some $a \in \operatorname{atoms}^{\mathscr{T}\operatorname{Src} f}$. Then there exist distinct points p, q such that $p, q \in \langle f\rangle^* a$. Thus $p\ (f \circ f^{-1})\ q$ and so $f \circ f^{-1} \not\sqsubseteq 1^{\mathbf{Rel}}_{\operatorname{Dst} f}$.

□

4.4. Product of typed sets

Definition 389. Product of typed sets is defined by the formula

$$(\mathscr{P}U, A) \times (\mathscr{P}W, B) = (U, W, A \times B).$$

Proposition 390. Product of typed sets is a **Rel**-morphism.

Proof. We need to prove $A \times B \subseteq U \times W$, but this is obvious. □

Obvious 391. Atoms of $\mathbf{Rel}(A, B)$ are exactly products $a \times b$ where a and b are atoms correspondingly of $\mathscr{T}A$ and $\mathscr{T}B$. $\mathbf{Rel}(A, B)$ is an atomistic poset.

Proposition 392. $f \not\asymp A \times B \Leftrightarrow A\ [f]^*\ B$ for every **Rel**-morphism f and $A \in \mathscr{T}\operatorname{Src} f$, $B \in \mathscr{T}\operatorname{Dst} f$.

Proof.

$$A\ [f]^*\ B \Leftrightarrow \exists x \in \operatorname{atoms} A, y \in \operatorname{atoms} B : x\ [f]^*\ y \Leftrightarrow$$
$$\exists x \in \operatorname{atoms}^{\mathscr{T}\operatorname{Src} f}, y \in \operatorname{atoms}^{\mathscr{T}\operatorname{Dst} f} : (x \times y \sqsubseteq f \wedge x \times y \sqsubseteq A \times B) \Leftrightarrow f \not\asymp A \times B.$$

□

Definition 393. *Image* and *domain* of a **Rel**-morphism f are typed sets defined by the formulas

$$\operatorname{dom}(U, W, f) = (\mathscr{P}U, \operatorname{dom} f) \quad \text{and} \quad \operatorname{im}(U, W, f) = (\mathscr{P}W, \operatorname{im} f).$$

Obvious 394. Image and domain of a **Rel**-morphism are really typed sets.

Definition 395. *Restriction* of a **Rel**-morphism to a typed set is defined by the formula $(U, W, f)|_{(\mathscr{P}U, X)} = (U, W, f|_X)$.

Obvious 396. Restriction of a **Rel**-morphism is **Rel**-morphism.

Obvious 397. $f|_A = f \sqcap (A \times \top^{\mathscr{T}\operatorname{Dst} f})$ for every **Rel**-morphism f and $A \in \mathscr{T}\operatorname{Src} f$.

Obvious 398. $\langle f\rangle^* X = \langle f\rangle^*(X \sqcap \operatorname{dom} f) = \operatorname{im}(f|_X)$ for every **Rel**-morphism f and $X \in \mathscr{T}\operatorname{Src} f$.

Obvious 399. $f \sqsubseteq A \times B \Leftrightarrow \operatorname{dom} f \sqsubseteq A \wedge \operatorname{im} f \sqsubseteq B$ for every **Rel**-morphism f and $A \in \mathscr{T}\operatorname{Src} f$, $B \in \mathscr{T}\operatorname{Dst} f$.

Theorem 400. Let A, B be sets. If $S \in \mathscr{P}(\mathscr{T}A \times \mathscr{T}B)$ then

$$\bigsqcap_{(A,B)\in S} (A \times B) = \bigsqcap \operatorname{dom} S \times \bigsqcap \operatorname{im} S.$$

Proof. For every atomic $x \in \mathscr{T}A$, $y \in \mathscr{T}B$ we have

$$x \times y \sqsubseteq \bigsqcap_{(A,B)\in S} (A \times B) \Leftrightarrow \forall (A,B) \in S : x \times y \sqsubseteq A \times B \Leftrightarrow$$
$$\forall (A,B) \in S : (x \sqsubseteq A \wedge y \sqsubseteq B) \Leftrightarrow \forall A \in \operatorname{dom} S : x \sqsubseteq A \wedge \forall B \in \operatorname{im} S : y \sqsubseteq B \Leftrightarrow$$
$$x \sqsubseteq \bigsqcap \operatorname{dom} S \wedge y \sqsubseteq \bigsqcap \operatorname{im} S \Leftrightarrow x \times y \sqsubseteq \bigsqcap \operatorname{dom} S \times \bigsqcap \operatorname{im} S.$$

□

Obvious 401. If U, W are sets and $A \in \mathscr{T}(U)$ then $A\times$ is a complete homomorphism from the lattice $\mathscr{T}(W)$ to the lattice $\mathbf{Rel}(U, W)$, if also $A \neq \bot$ then it is an order embedding.

CHAPTER 5

Filters and filtrators

This chapter is based on my article [32].

This chapter is grouped in the following way:

- First it goes a short introduction in pedagogical order (first less general stuff and examples, last the most general stuff):
  - filters on a set;
  - filters on a meet-semilattice;
  - filters on a poset.
- Then it goes the formal part.

5.1. Implication tuples

Definition 402. An *implications tuple* is a tuple $(P_1, \dots, P_n)$ such that $P_1 \Rightarrow \dots \Rightarrow P_n$.

Obvious 403. $(P_1, \dots, P_n)$ is an implications tuple iff $P_i \Rightarrow P_j$ for every $i < j$ (where $i, j \in \{1, \dots, n\}$).

The following is an example of a theorem using an implication tuple:

Example 404. The following is an implications tuple:

1°. A.
2°. B.
3°. C.

This example means just that $A \Rightarrow B \Rightarrow C$.

I prefer here a verbal description instead of symbolic implications $A \Rightarrow B \Rightarrow C$, because A, B, C may be long English phrases and they may not fit into the formula layout.

The main (intuitive) idea of the theorem is expressed by the implication $P_1 \Rightarrow P_n$, the rest implications ($P_2 \Rightarrow P_n$, $P_3 \Rightarrow P_n$, ...) are purely technical, as they express generalizations of the main idea.

For uniformity theorems in the section about filters and filtrators start with the same P_1: "$(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator." (defined below) That means that the main idea of the theorem is about powerset filtrators, the rest implications (like $P_2 \Rightarrow P_n$, $P_3 \Rightarrow P_n$, ...) are just technical generalizations.

5.2. Introduction to filters and filtrators

5.2.1. Filters on a set. We sometimes want to define something resembling an infinitely small (or infinitely big) set, for example the infinitely small interval near 0 on the real line. Of course there is no such set, just like as there is no natural number which is the difference $2 - 3$. To overcome this shortcoming we introduce whole numbers, and $2 - 3$ becomes well defined. In the same way to consider things which are like infinitely small (or infinitely big) sets we introduce *filters*.

An example of a filter is the infinitely small interval near 0 on the real line. To come to infinitely small, we consider all intervals $]-\epsilon; \epsilon[$ for all $\epsilon > 0$. This filter

consists of all intervals $]-\epsilon;\epsilon[$ for all $\epsilon > 0$ and also all subsets of $\mathbb{R}$ containing such intervals as subsets. Informally speaking, this is the greatest filter contained in every interval $]-\epsilon;\epsilon[$ for all $\epsilon > 0$.

Definition 405. A filter on a set $\mho$ is a $\mathcal{F} \in \mathscr{P}\mathscr{P}\mho$ such that:

1°. $\forall A, B \in \mathcal{F} : A \cap B \in \mathcal{F}$;
2°. $\forall A, B \in \mathscr{P}\mho : (A \in \mathcal{F} \wedge B \supseteq A \Rightarrow B \in \mathcal{F})$.

Exercise 406. Verify that the above introduced infinitely small interval near 0 on the real line is a filter on $\mathbb{R}$.

Exercise 407. Describe "the neighborhood of positive infinity" filter on $\mathbb{R}$.

Definition 408. A filter not containing empty set is called a *proper filter*.

Obvious 409. The non-proper filter is $\mathscr{P}\mho$.

Remark 410. Some other authors require that all filters are proper. This is a stupid idea and we allow non-proper filters, in the same way as we allow to use the number 0.

5.2.2. Intro to filters on a meet-semilattice. A trivial generalization of the above:

Definition 411. A filter on a meet-semilattice $\mathfrak{Z}$ is a $\mathcal{F} \in \mathscr{P}\mathfrak{Z}$ such that:

1°. $\forall A, B \in \mathcal{F} : A \sqcap B \in \mathcal{F}$;
2°. $\forall A, B \in \mathfrak{Z} : (A \in \mathcal{F} \wedge B \sqsupseteq A \Rightarrow B \in \mathcal{F})$.

5.2.3. Intro to filters on a poset.

Definition 412. A filter on a poset $\mathfrak{Z}$ is a $\mathcal{F} \in \mathscr{P}\mathfrak{Z}$ such that:

1°. $\forall A, B \in \mathcal{F} \exists C \in \mathcal{F} : C \sqsubseteq A, B$;
2°. $\forall A, B \in \mathfrak{Z} : (A \in \mathcal{F} \wedge B \sqsupseteq A \Rightarrow B \in \mathcal{F})$.

It is easy to show (and there is a proof of it somewhere below) that this coincides with the above definition in the case if $\mathfrak{Z}$ is a meet-semilattice.

5.3. Filters on a poset

5.3.1. Filters on posets. Let $\mathfrak{Z}$ be a poset.

Definition 413. *Filter base* is a nonempty subset F of $\mathfrak{Z}$ such that

$$\forall X, Y \in F \exists Z \in F : (Z \sqsubseteq X \wedge Z \sqsubseteq Y).$$

Definition 414. *Ideal base* is a nonempty subset F of $\mathfrak{Z}$ such that

$$\forall X, Y \in F \exists Z \in F : (Z \sqsupseteq X \wedge Z \sqsupseteq Y).$$

Obvious 415. Ideal base is the dual of filter base.

Obvious 416.

1°. A poset with a lowest element is a filter base.
2°. A poset with a greatest element is an ideal base.

Obvious 417.

1°. A meet-semilattice is a filter base.
2°. A join-semilattice is an ideal base.

Obvious 418. A nonempty chain is a filter base and an ideal base.

Definition 419. *Filter* is a subset of $\mathfrak{Z}$ which is both a filter base and an upper set.

I will denote the set of filters (for a given or implied poset $\mathfrak{Z}$) as $\mathfrak{F}$ and call $\mathfrak{F}$ the set of filters over the poset $\mathfrak{Z}$.

PROPOSITION 420. If $\top$ is the maximal element of $\mathfrak{Z}$ then $\top \in F$ for every filter F.

PROOF. If $\top \notin F$ then $\forall K \in \mathfrak{Z} : K \notin F$ and so F is empty what is impossible. □

PROPOSITION 421. Let S be a filter base on a poset. If $A_0, \dots, A_n \in S$ ($n \in \mathbb{N}$), then

$$\exists C \in S : (C \sqsubseteq A_0 \wedge \ldots \wedge C \sqsubseteq A_n).$$

PROOF. It can be easily proved by induction. □

DEFINITION 422. A function f from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ *preserves filtered meets* iff whenever $\bigsqcap S$ is defined for a filter base S on $\mathfrak{A}$ we have $f \bigsqcap S = \bigsqcap \langle f \rangle^* S$.

5.3.2. Filters on meet-semilattices.

THEOREM 423. If $\mathfrak{Z}$ is a meet-semilattice and F is a nonempty subset of $\mathfrak{Z}$ then the following conditions are equivalent:

1°. F is a filter.
2°. $\forall X, Y \in F : X \sqcap Y \in F$ and F is an upper set.
3°. $\forall X, Y \in \mathfrak{Z} : (X, Y \in F \Leftrightarrow X \sqcap Y \in F)$.

PROOF.

1°⇒2°. Let F be a filter. Then F is an upper set. If $X, Y \in F$ then $Z \sqsubseteq X \wedge Z \sqsubseteq Y$ for some $Z \in F$. Because F is an upper set and $Z \sqsubseteq X \sqcap Y$ then $X \sqcap Y \in F$.

2°⇒1°. Let $\forall X, Y \in F : X \sqcap Y \in F$ and F be an upper set. We need to prove that F is a filter base. But it is obvious taking $Z = X \sqcap Y$ (we have also taken into account that $F \neq \emptyset$).

2°⇒3°. Let $\forall X, Y \in F : X \sqcap Y \in F$ and F be an upper set. Then

$$\forall X, Y \in \mathfrak{Z} : (X, Y \in F \Rightarrow X \sqcap Y \in F).$$

Let $X \sqcap Y \in F$; then $X, Y \in F$ because F is an upper set.

3°⇒2°. Let

$$\forall X, Y \in \mathfrak{Z} : (X, Y \in F \Leftrightarrow X \sqcap Y \in F).$$

Then $\forall X, Y \in F : X \sqcap Y \in F$. Let $X \in F$ and $X \sqsubseteq Y \in \mathfrak{Z}$. Then $X \sqcap Y = X \in F$. Consequently $X, Y \in F$. So F is an upper set.

□

PROPOSITION 424. Let S be a filter base on a meet-semilattice. If $A_0, \dots, A_n \in S$ ($n \in \mathbb{N}$), then

$$\exists C \in S : C \sqsubseteq A_0 \sqcap \cdots \sqcap A_n.$$

PROOF. It can be easily proved by induction. □

PROPOSITION 425. If $\mathfrak{Z}$ is a meet-semilattice and S is a filter base on it, $A \in \mathfrak{Z}$, then $\langle A \sqcap \rangle^* S$ is also a filter base.

PROOF. $\langle A \sqcap \rangle^* S \neq \emptyset$ because $S \neq \emptyset$.

Let $X, Y \in \langle A \sqcap \rangle^* S$. Then $X = A \sqcap X'$ and $Y = A \sqcap Y'$ where $X', Y' \in S$. There exists $Z' \in S$ such that $Z' \sqsubseteq X' \sqcap Y'$. So $X \sqcap Y = A \sqcap X' \sqcap Y' \sqsupseteq A \sqcap Z' \in \langle A \sqcap \rangle^* S$. □

5.3.3. Order of filters. Principal filters. I will make the set of filters $\mathfrak{F}$ into a poset by the order defined by the formula: $a \sqsubseteq b \Leftrightarrow a \supseteq b$.

DEFINITION 426. The principal filter corresponding to an element $a \in \mathfrak{Z}$ is

$$\uparrow a = \left\{ \frac{x \in \mathfrak{Z}}{x \sqsupseteq a} \right\}.$$

Elements of $\mathfrak{P} = \langle\uparrow\rangle^* \mathfrak{Z}$ are called *principal filters*.

OBVIOUS 427. Principal filters are filters.

OBVIOUS 428. $\uparrow$ is an order embedding from $\mathfrak{Z}$ to $\mathfrak{F}$.

COROLLARY 429. $\uparrow$ is an order isomorphism between $\mathfrak{Z}$ and $\mathfrak{P}$.

We will equate principal filters with corresponding elements of the base poset (in the same way as we equate for example nonnegative whole numbers and natural numbers).

PROPOSITION 430. $\uparrow K \sqsupseteq \mathcal{A} \Leftrightarrow K \in \mathcal{A}$.

PROOF. $\uparrow K \sqsupseteq \mathcal{A} \Leftrightarrow \uparrow K \subseteq \mathcal{A} \Leftrightarrow K \in \mathcal{A}$. □

5.4. Filters on a Set

Consider filters on the poset $\mathfrak{Z} = \mathscr{P}\mathfrak{U}$ (where $\mathfrak{U}$ is some fixed set) with the order $A \sqsubseteq B \Leftrightarrow A \subseteq B$ (for $A, B \in \mathscr{P}\mathfrak{A}$).

In fact, it is a complete atomistic boolean lattice with $\bigsqcap S = \bigcap S$, $\bigsqcup S = \bigcup S$, $\overline{A} = \mathfrak{U} \setminus A$ for every $S \in \mathscr{P}\mathscr{P}\mathfrak{U}$ and $A \in \mathscr{P}\mathfrak{U}$, atoms being one-element sets.

DEFINITION 431. I will call a filter on the lattice of all subsets of a given set $\mathfrak{U}$ as a *filter on set*.

DEFINITION 432. I will denote the set on which a filter $\mathcal{F}$ is defined as $\operatorname{Base}(\mathcal{F})$.

OBVIOUS 433. $\operatorname{Base}(\mathcal{F}) = \bigcup \mathcal{F}$.

PROPOSITION 434. The following are equivalent for a non-empty set $F \in \mathscr{P}\mathscr{P}\mathfrak{U}$:

1°. F is a filter.
2°. $\forall X, Y \in F : X \cap Y \in F$ and F is an upper set.
3°. $\forall X, Y \in \mathscr{P}\mathfrak{U} : (X, Y \in F \Leftrightarrow X \cap Y \in F)$.

PROOF. By theorem 423. □

OBVIOUS 435. The minimal filter on $\mathscr{P}\mathfrak{U}$ is $\mathscr{P}\mathfrak{U}$.

OBVIOUS 436. The maximal filter on $\mathscr{P}\mathfrak{U}$ is $\{\mathfrak{U}\}$.

I will denote $\uparrow A = \uparrow^{\mathfrak{U}} A = \uparrow^{\mathscr{P}\mathfrak{U}} A$. (The distinction between conflicting notations $\uparrow^{\mathfrak{U}} A$ and $\uparrow^{\mathscr{P}\mathfrak{U}} A$ will be clear from the context.)

PROPOSITION 437. Every filter on a finite set is principal.

PROOF. Let $\mathcal{F}$ be a filter on a finite set. Then obviously $\mathcal{F} = \bigsqcap^{3} \operatorname{up} \mathcal{F}$ and thus $\mathcal{F}$ is principal. □

5.5. Filtrators

$(\mathfrak{F}, \mathfrak{P})$ is a poset and its subset (with induced order on the subset). I call pairs of a poset and its subset like this *filtrators*.

Definition 438. I will call a *filtrator* a pair $(\mathfrak{A}, \mathfrak{Z})$ of a poset $\mathfrak{A}$ and its subset $\mathfrak{Z} \subseteq \mathfrak{A}$. I call $\mathfrak{A}$ the *base* of the filtrator and $\mathfrak{Z}$ the *core* of the filtrator. I will also say that $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator *over* poset $\mathfrak{Z}$.

I will denote $\operatorname{base}(\mathfrak{A}, \mathfrak{Z}) = \mathfrak{A}$, $\operatorname{core}(\mathfrak{A}, \mathfrak{Z}) = \mathfrak{Z}$ for a filtrator $(\mathfrak{A}, \mathfrak{Z})$.

While *filters* are customary and well known mathematical objects, the concept of *filtrators* is probably first researched by me.

When speaking about filters, we will imply that we consider the filtrator $(\mathfrak{F}, \mathfrak{P})$ or what is the same (as we equate principal filters with base elements) the filtrator $(\mathfrak{F}, \mathfrak{Z})$.

Definition 439. I will call a *lattice filtrator* a pair $(\mathfrak{A}, \mathfrak{Z})$ of a lattice $\mathfrak{A}$ and its subset $\mathfrak{Z} \subseteq \mathfrak{A}$.

Definition 440. I will call a *complete lattice filtrator* a pair $(\mathfrak{A}, \mathfrak{Z})$ of a complete lattice $\mathfrak{A}$ and its subset $\mathfrak{Z} \subseteq \mathfrak{A}$.

Definition 441. I will call a *central filtrator* a filtrator $(\mathfrak{A}, Z(\mathfrak{A}))$ where $Z(\mathfrak{A})$ is the center of a bounded lattice $\mathfrak{A}$.

Definition 442. I will call *element* of a filtrator an element of its base.

Definition 443. $\operatorname{up}^{\mathfrak{Z}} a = \operatorname{up} a = \left\{ \frac{c \in \mathfrak{Z}}{c \sqsupseteq a} \right\}$ for an element a of a filtrator.

Definition 444. $\operatorname{down}^{\mathfrak{Z}} a = \operatorname{down} a = \left\{ \frac{c \in \mathfrak{Z}}{c \sqsubseteq a} \right\}$ for an element a of a filtrator.

Obvious 445. "up" and "down" are dual.

Our main purpose here is knowing properties of the core of a filtrator to infer properties of the base of the filtrator, specifically properties of $\operatorname{up} a$ for every element a.

Definition 446. I call a filtrator *with join-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $\bigsqcup^{\mathfrak{Z}} S = \bigsqcup^{\mathfrak{A}} S$ whenever $\bigsqcup^{\mathfrak{Z}} S$ exists for $S \in \mathscr{P}\mathfrak{Z}$.

Definition 447. I call a filtrator *with meet-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $\bigsqcap^{\mathfrak{Z}} S = \bigsqcap^{\mathfrak{A}} S$ whenever $\bigsqcap^{\mathfrak{Z}} S$ exists for $S \in \mathscr{P}\mathfrak{Z}$.

Definition 448. I call a filtrator with *binarily join-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $a \sqcup^{\mathfrak{Z}} b = a \sqcup^{\mathfrak{A}} b$ whenever $a \sqcup^{\mathfrak{Z}} b$ exists for $a, b \in \mathfrak{Z}$.

Definition 449. I call a filtrator with *binarily meet-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $a \sqcap^{\mathfrak{Z}} b = a \sqcap^{\mathfrak{A}} b$ whenever $a \sqcap^{\mathfrak{Z}} b$ exists for $a, b \in \mathfrak{Z}$.

Definition 450. *Prefiltered filtrator* is a filtrator $(\mathfrak{A}, \mathfrak{Z})$ such that "up" is injective.

Definition 451. *Filtered filtrator* is a filtrator $(\mathfrak{A}, \mathfrak{Z})$ such that

$$\forall a, b \in \mathfrak{A} : (\operatorname{up} a \supseteq \operatorname{up} b \Rightarrow a \sqsubseteq b).$$

Theorem 452. A filtrator $(\mathfrak{A}, \mathfrak{Z})$ is filtered iff $\forall a \in \mathfrak{A} : a = \bigsqcap^{\mathfrak{A}} \operatorname{up} a$.

Proof.

$\Leftarrow$. $\operatorname{up} a \supseteq \operatorname{up} b \Rightarrow \bigsqcap^{\mathfrak{A}} \operatorname{up} a \sqsubseteq \bigsqcap^{\mathfrak{A}} \operatorname{up} b \Rightarrow a \sqsubseteq b$.

$\Rightarrow$. $a = \bigsqcap^{\mathfrak{A}} \operatorname{up} a$ is equivalent to a is a greatest lower bound of $\operatorname{up} a$. That is the implication that b is lower bound of $\operatorname{up} a$ implies $a \sqsupseteq b$.

b is lower bound of $\operatorname{up} a$ implies $\operatorname{up} b \supseteq \operatorname{up} a$. So as it is filtered $a \sqsupseteq b$.

□

Obvious 453. Every filtered filtrator is prefiltered.

Obvious 454. "up" is a straight map from $\mathfrak{A}$ to the dual of the poset $\mathscr{P}\mathfrak{Z}$ if $(\mathfrak{A}, \mathfrak{Z})$ is a filtered filtrator.

Definition 455. An *isomorphism* between filtrators $(\mathfrak{A}_0, \mathfrak{Z}_0)$ and $(\mathfrak{A}_1, \mathfrak{Z}_1)$ is an isomorphism between posets $\mathfrak{A}_0$ and $\mathfrak{A}_1$ such that it maps $\mathfrak{Z}_0$ into $\mathfrak{Z}_1$.

Obvious 456. Isomorphism isomorphically maps the order on $\mathfrak{Z}_0$ into order on $\mathfrak{Z}_1$.

Definition 457. Two filtrators are *isomorphic* when there exists an isomorphism between them.

Definition 458. I will call *primary filtrator* a filtrator isomorphic to the filtrator consisting of the set of filters on a poset and the set of principal filters on this poset.

Obvious 459. The order on a primary filtrator is defined by the formula $a \sqsubseteq b \Leftrightarrow \operatorname{up} a \supseteq \operatorname{up} b$.

Definition 460. I will call a primary filtrator over a poset isomorphic to a powerset as *powerset filtrator*.

Obvious 461. $\operatorname{up} \mathcal{F}$ is a filter for every element $\mathcal{F}$ of a primary filtrator. Reversely, there exists a filter $\mathcal{F}$ if $\operatorname{up} \mathcal{F}$ is a filter.

Theorem 462. For every poset $\mathfrak{Z}$ there exists a poset $\mathfrak{A} \supseteq \mathfrak{Z}$ such that $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.

Proof. See appendix A. □

5.5.1. Filtrators with Separable Core.

Definition 463. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. It is a *filtrator with separable core* when

$$\forall x, y \in \mathfrak{A} : (x \asymp^{\mathfrak{A}} y \Rightarrow \exists X \in \operatorname{up} x : X \asymp^{\mathfrak{A}} y).$$

Proposition 464. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. It is a *filtrator with separable core* iff

$$\forall x, y \in \mathfrak{A} : (x \asymp^{\mathfrak{A}} y \Rightarrow \exists X \in \operatorname{up} x, Y \in \operatorname{up} y : X \asymp^{\mathfrak{A}} Y).$$

Proof.

$\Rightarrow$. Apply the definition twice.

$\Leftarrow$. Obvious.

□

Definition 465. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. It is a *filtrator with co-separable core* when

$$\forall x, y \in \mathfrak{A} : (x \equiv^{\mathfrak{A}} y \Rightarrow \exists X \in \operatorname{down} x : X \equiv^{\mathfrak{A}} y).$$

Obvious 466. Co-separability is the dual of separability.

Definition 467. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. It is a *filtrator with co-separable core* when

$$\forall x, y \in \mathfrak{A} : (x \equiv^{\mathfrak{A}} y \Rightarrow \exists X \in \operatorname{down} x, Y \in \operatorname{down} y : X \equiv^{\mathfrak{A}} Y).$$

Proof. By duality. □

5.6. Alternative primary filtrators

5.6.1. Lemmas.

LEMMA 468. A set F is a lower set iff $\overline{F}$ is an upper set.

PROOF. $X \in \overline{F} \wedge Z \sqsupseteq X \Rightarrow Z \in \overline{F}$ is equivalent to $Z \in F \Rightarrow X \in F \vee Z \not\sqsupseteq X$ is equivalent $Z \in F \Rightarrow (Z \sqsupseteq X \Rightarrow X \in F)$ is equivalent $Z \in F \wedge X \sqsubseteq Z \Rightarrow X \in F$. □

PROPOSITION 469. Let $\mathfrak{Z}$ be a poset with least element $\perp$. Then for upper set F we have $F \neq \mathscr{P}\mathfrak{Z} \Leftrightarrow \perp \notin F$.

PROOF.

$\Rightarrow$. If $\perp \in F$ then $F = \mathscr{P}\mathfrak{Z}$ because F is an upper set.
$\Leftarrow$. Obvious.

□

5.6.2. Informal introduction. We have already defined filters on a poset. Now we will define three other sets which are order-isomorphic to the set of filters on a poset: ideals ($\mathfrak{I}$), free stars ($\mathfrak{S}$), and mixers ($\mathfrak{M}$).

These four kinds of objects are related through commutative diagrams. First we will paint an informal commutative diagram (it makes no formal sense because it is not pointed the poset for which the filters are defined):

$$\begin{tikzcd}
\mathfrak{F} \arrow[r, leftrightarrow, "\langle\text{dual}\rangle^*"] \arrow[d, leftrightarrow, "\neg"] & \mathfrak{I} \arrow[d, leftrightarrow, "\neg"] \\
\mathfrak{M} \arrow[r, leftrightarrow, "\langle\text{dual}\rangle^*"] & \mathfrak{S}
\end{tikzcd}$$

Then we can define ideals, free stars, and mixers as sets following certain formulas. You can check that the intuition behind these formulas follows the above commutative diagram. (That is transforming these formulas by the course of the above diagram, you get formulas of the other objects in this list.)

After this, we will paint some formal commutative diagrams similar to the above diagram but with particular posets at which filters, ideals, free stars, and mixers are defined.

5.6.3. Definitions of ideals, free stars, and mixers. *Filters* and *ideals* are well known concepts. The terms *free stars* and *mixers* are my new terminology.

Recall that *filters* are nonempty sets F with $A, B \in F \Leftrightarrow \exists Z \in F : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

DEFINITION 470. *Ideals* are nonempty sets F with $A, B \in F \Leftrightarrow \exists Z \in F : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).

DEFINITION 471. *Free stars* are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Leftrightarrow \exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).

DEFINITION 472. *Mixers* are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Leftrightarrow \exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

By duality and and an above theorem about filters, we have:

PROPOSITION 473.

- Filters are nonempty upper sets F with $A, B \in F \Rightarrow \exists Z \in F : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Ideals are nonempty lower sets F with $A, B \in F \Rightarrow \exists Z \in F : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).

- Free stars are upper sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Rightarrow \exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Mixers are lower sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Rightarrow \exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

PROPOSITION 474. The following are equivalent:

1°. F is a free star.
2°. $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F) \Leftrightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and $F \neq \mathscr{P}\mathfrak{Z}$.
3°. $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F) \Rightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and F is an upper set and $F \neq \mathscr{P}\mathfrak{Z}$.

PROOF.

1°⇔2°. The following is a chain of equivalencies:

$$\exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B) \Leftrightarrow A \notin F \wedge B \notin F;$$
$$\forall Z \in \overline{F} : \neg(Z \sqsupseteq A \wedge Z \sqsupseteq B) \Leftrightarrow A \in F \vee B \in F;$$
$$\forall Z \in \mathfrak{Z} : (Z \notin F \Rightarrow \neg(Z \sqsupseteq A \wedge Z \sqsupseteq B)) \Leftrightarrow A \in F \vee B \in F;$$
$$\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F) \Leftrightarrow A \in F \vee B \in F.$$

2°⇒3°. Let $A = B \in F$. Then $A \in F \vee B \in F$. So $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F)$ that is $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \Rightarrow Z \in F)$ that is F is an upper set.

3°⇒2°. We need to prove that F is an upper set. let $A \in F$ and $A \sqsubseteq B \in \mathfrak{Z}$. Then $A \in F \vee B \in F$ and thus $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F)$ that is $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq B \Rightarrow Z \in F)$ and so $B \in F$.

□

COROLLARY 475. The following are equivalent:

1°. F is a mixer.
2°. $\forall Z \in \mathfrak{Z} : (Z \sqsubseteq A \wedge Z \sqsubseteq B \Rightarrow Z \in F) \Leftrightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and $F \neq \mathscr{P}\mathfrak{Z}$.
3°. $\forall Z \in \mathfrak{Z} : (Z \sqsubseteq A \wedge Z \sqsubseteq B \Rightarrow Z \in F) \Rightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and F is an lower set and $F \neq \mathscr{P}\mathfrak{Z}$.

OBVIOUS 476.

1°. A free star cannot contain the least element of the poset.
2°. A mixer cannot contain the greatest element of the poset.

5.6.4. Filters, ideals, free stars, and mixers on semilattices.

PROPOSITION 477.

- Free stars are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftrightarrow \neg\exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Free stars are upper sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftarrow \neg\exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Mixers are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftrightarrow \neg\exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Mixers are lower sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftarrow \neg\exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

PROOF. By duality. □

By duality and and an above theorem about filters, we have:

PROPOSITION 478.

- Filters are nonempty sets F with $A \sqcap B \in F \Leftrightarrow A \in F \wedge B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a meet-semilattice.
- Ideals are nonempty sets F with $A \sqcup B \in F \Leftrightarrow A \in F \wedge B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a join-semilattice.
- Free stars are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \sqcup B \in F \Leftrightarrow A \in F \vee B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a join-semilattice.
- Mixers are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \sqcap B \in F \Leftrightarrow A \in F \vee B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a meet-semilattice.

By duality and and an above theorem about filters, we have:

Proposition 479.

- Filters are nonempty upper sets F with $A \sqcap B \in F \Leftarrow A \in F \wedge B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a meet-semilattice.
- Ideals are nonempty lower sets F with $A \sqcup B \in F \Leftarrow A \in F \wedge B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a join-semilattice.
- Free stars are upper sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \sqcup B \in F \Rightarrow A \in F \vee B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a join-semilattice.
- Mixers are lower sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \sqcap B \in F \Rightarrow A \in F \vee B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a meet-semilattice.

5.6.5. The general diagram. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two posets connected by an order reversing isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$. We have commutative diagram on the figure 3 in the category **Set**:

$$\begin{tikzcd}
\mathscr{P}\mathfrak{A} \arrow[r, shift left, "\langle\theta\rangle^*"] \arrow[d, leftrightarrow, "\neg"] & \mathscr{P}\mathfrak{B} \arrow[l, shift left, "\langle\theta^{-1}\rangle^*"] \arrow[d, leftrightarrow, "\neg"] \\
\mathscr{P}\mathfrak{A} \arrow[r, shift left, "\langle\theta\rangle^*"] & \mathscr{P}\mathfrak{B} \arrow[l, shift left, "\langle\theta^{-1}\rangle^*"]
\end{tikzcd}$$

Figure 3

Theorem 480. This diagram is commutative, every arrow of this diagram is an isomorphism, every cycle in this diagrams is an identity (therefore "parallel" arrows are mutually inverse).

Proof. That every arrow is an isomorphism is obvious.
Show that $\langle\theta\rangle^* \neg X = \neg \langle\theta\rangle^* X$ for every set $X \in \mathscr{P}\mathfrak{A}$.
Really,

$$p \in \langle\theta\rangle^* \neg X \Leftrightarrow \exists q \in \neg X : p = \theta q \Leftrightarrow \exists q \in \neg X : \theta^{-1} p = q \Leftrightarrow \theta^{-1} p \in \neg X \Leftrightarrow$$
$$\nexists q \in X : q = \theta^{-1} p \Leftrightarrow \nexists q \in X : \theta q = p \Leftrightarrow p \notin \langle\theta\rangle^* X \Leftrightarrow p \in \neg \langle\theta\rangle^* X.$$

Thus the theorem follows from lemma 197. □

This diagram can be restricted to filters, ideals, free stars, and mixers, see figure 4:

Theorem 481. It is a restriction of the above diagram. Every arrow of this diagram is an isomorphism, every cycle in these diagrams is an identity. (To prove that, is an easy application of duality and the above lemma.)

$$\begin{tikzcd}
\mathfrak{F}(\mathfrak{A}) \arrow[r, shift left, "\langle\theta\rangle^*"] \arrow[d, leftrightarrow, "\neg"] & \mathfrak{I}(\mathfrak{B}) \arrow[l, shift left, "\langle\theta^{-1}\rangle^*"] \arrow[d, leftrightarrow, "\neg"] \\
\mathfrak{M}(\mathfrak{A}) \arrow[r, shift left, "\langle\theta\rangle^*"] & \mathfrak{S}(\mathfrak{B}) \arrow[l, shift left, "\langle\theta^{-1}\rangle^*"]
\end{tikzcd}$$

FIGURE 4

5.6.6. Special diagrams. Here are two important special cases of the above diagram:

$$\begin{tikzcd}
\mathfrak{F}(\mathfrak{A}) \arrow[r, leftrightarrow, "\langle\mathrm{dual}\rangle^*"] \arrow[d, leftrightarrow, "\neg"] & \mathfrak{I}(\mathrm{dual}\,\mathfrak{A}) \arrow[d, leftrightarrow, "\neg"] \\
\mathfrak{M}(\mathfrak{A}) \arrow[r, leftrightarrow, "\langle\mathrm{dual}\rangle^*"] & \mathfrak{S}(\mathrm{dual}\,\mathfrak{A})
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathfrak{F}(\mathfrak{A}) \arrow[r, leftrightarrow, "\langle\neg\rangle^*"] \arrow[d, leftrightarrow, "\neg"] & \mathfrak{I}(\mathfrak{A}) \arrow[d, leftrightarrow, "\neg"] \\
\mathfrak{M}(\mathfrak{A}) \arrow[r, leftrightarrow, "\langle\neg\rangle^*"] & \mathfrak{S}(\mathfrak{A})
\end{tikzcd} \tag{1}$$

(the second diagram is defined for a boolean lattice $\mathfrak{A}$).

5.6.7. Order of ideals, free stars, mixers. Define order of ideals, free stars, mixers in such a way that the above diagrams isomorphically preserve order of filters:

- $A \sqsubseteq B \Leftrightarrow A \supseteq B$ for filters and ideals;
- $A \sqsubseteq B \Leftrightarrow A \subseteq B$ for free stars and mixers.

5.6.8. Principal ideals, free stars, mixers.

DEFINITION 482. *Principal* ideal generated by an element a of poset $\mathfrak{A}$ is $\downarrow a = \left\{ \frac{x \in \mathfrak{A}}{x \sqsubseteq a} \right\}$.

DEFINITION 483. An ideal is *principal* iff it is generated by some poset element.

DEFINITION 484. The *filtrator of ideals* on a given poset is the pair consisting of the set of ideals and the set of principal ideals.

The above poset isomorphism maps principal filters into principal ideals and thus is an isomorphism between the filtrator of filters on a poset and the filtrator of ideals on the dual poset.

EXERCISE 485. Define principal free stars and mixers, filtrators of free stars and mixers and isomorphisms of these with the filtrator of filters (these isomorphisms exist because the posets of free stars and mixers are isomorphic to the poset of filters).

OBVIOUS 486. The following filtrators are primary:

- filtrators of filters;
- filtrators of ideals;
- filtrators of free stars;
- filtrators of mixers.

5.6.8.1. *Principal free stars.*

PROPOSITION 487. An upper set $F \in \mathscr{P}\mathfrak{Z}$ is a principal filter iff $\exists Z \in F \forall P \in F : Z \sqsubseteq P$.

PROOF.

$\Rightarrow$. Obvious.

$\Leftarrow$. Let $Z \in F$ and $\forall P \in F : Z \sqsubseteq P$. F is nonempty because $Z \in F$. It remains to prove that $Z \sqsubseteq P \Leftrightarrow P \in F$. The reverse implication follows from $\forall P \in F : Z \sqsubseteq P$. The direct implication follows from that F is an upper set.

□

LEMMA 488. If $S \in \mathscr{P}\mathfrak{Z}$ is not the complement of empty set and for every $T \in \mathscr{P}\mathfrak{Z}$

$$\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Leftrightarrow T \cap S \neq \emptyset,$$

then S is a free star.

PROOF. Take $T = \{A, B\}$. Then $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in S) \Leftrightarrow A \in S \vee B \in S$. So S is a free star. □

PROPOSITION 489. A set $S \in \mathscr{P}\mathfrak{Z}$ is a principal free star iff S is not the complement of empty set and for every $T \in \mathscr{P}\mathfrak{Z}$

$$\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Leftrightarrow T \cap S \neq \emptyset.$$

PROOF. Let $S = \overline{\langle \mathrm{dual} \rangle^* F}$. We need to prove that F is a principal filter iff the above formula holds. Really, we have the following chain of equivalencies:

$\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Leftrightarrow T \cap S \neq \emptyset$;
$\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \notin \langle \mathrm{dual} \rangle^* F) \Leftrightarrow T \cap \overline{\langle \mathrm{dual} \rangle^* F} \neq \emptyset$;
$\forall Z \in \mathrm{dual}\, \mathfrak{Z} : (\forall X \in T : Z \sqsubseteq X \Rightarrow Z \notin F) \Leftrightarrow T \cap \overline{F} \neq \emptyset$;
$\forall Z \in \mathrm{dual}\, \mathfrak{Z} : (\forall X \in T : Z \sqsubseteq X \Rightarrow Z \notin F) \Leftrightarrow T \nsubseteq F$;
$T \subseteq F \Leftrightarrow \neg \forall Z \in \mathrm{dual}\, \mathfrak{Z} : (Z \in F \Rightarrow \neg \forall X \in T : Z \sqsubseteq X)$;
$T \subseteq F \Leftrightarrow \neg \forall Z \in \mathrm{dual}\, \mathfrak{Z} : (Z \notin F \vee \neg \forall X \in T : Z \sqsubseteq X)$;
$T \subseteq F \Leftrightarrow \exists Z \in \mathrm{dual}\, \mathfrak{Z} : (Z \in F \wedge \forall X \in T : Z \sqsubseteq X)$;
$T \subseteq F \Leftrightarrow \exists Z \in F \forall X \in T : Z \sqsubseteq X$;
$\exists Z \in F \forall X \in F : Z \sqsubseteq X$ that is F is a principal filter (S is an upper set because by the lemma it is a free star; thus F is also an upper set). □

PROPOSITION 490. $S \in \mathscr{P}\mathfrak{Z}$ where $\mathfrak{Z}$ is a poset is a principal free star iff all the following:

1°. The least element (if it exists) is not in S.
2°. $\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Rightarrow T \cap S \neq \emptyset$ for every $T \in \mathscr{P}\mathfrak{Z}$.
3°. S is an upper set.

PROOF.

$\Rightarrow$. 1° and 2° are obvious. S is an upper set because S is a free star.
$\Leftarrow$. We need to prove that

$$\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Leftarrow T \cap S \neq \emptyset.$$

Let $X' \in T \cap S$. Then $\forall X \in T : Z \sqsupseteq X \Rightarrow Z \sqsupseteq X' \Rightarrow Z \in S$ because S is an upper set.

□

PROPOSITION 491. Let $\mathfrak{Z}$ be a complete lattice. $S \in \mathscr{P}\mathfrak{Z}$ is a principal free star iff all the following:

1°. The least element is not in S.
2°. $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$ for every $T \in \mathscr{P}\mathfrak{Z}$.
3°. S is an upper set.

PROOF.

$\Rightarrow$. We need to prove only $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$. Let $\bigsqcup T \in S$. Because S is an upper set, we have $\forall X \in T : Z \sqsupseteq X \Rightarrow Z \sqsupseteq \bigsqcup T \Rightarrow Z \in S$ for every $Z \in \mathfrak{Z}$; from which we conclude $T \cap S \neq \emptyset$.

$\Leftarrow$. We need to prove only $\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Rightarrow T \cap S \neq \emptyset$.

Really, if $\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S)$ then $\bigsqcup T \in S$ and thus $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$.

□

Proposition 492. Let $\mathfrak{Z}$ be a complete lattice. $S \in \mathscr{P}\mathfrak{Z}$ is a principal free star iff the least element is not in S and for every $T \in \mathscr{P}\mathfrak{Z}$

$$\bigsqcup T \in S \Leftrightarrow T \cap S \neq \emptyset.$$

Proof.

$\Rightarrow$. We need to prove only $\bigsqcup T \in S \Leftarrow T \cap S \neq \emptyset$ what follows from that S is an upper set.

$\Leftarrow$. We need to prove only that S is an upper set. To prove this we can use the fact that S is a free star.

□

Exercise 493. Write down similar formulas for mixers.

5.6.9. Starrish posets.

Definition 494. I will call a poset *starrish* when the full star $\star a$ is a free star for every element a of this poset.

Proposition 495. Every distributive lattice is starrish.

Proof. Let $\mathfrak{A}$ be a distributive lattice, $a \in \mathfrak{A}$. Obviously $\bot \notin \star a$ (if $\bot$ exists); obviously $\star a$ is an upper set. If $x \sqcup y \in \star a$, then $(x \sqcup y) \sqcap a$ is non-least that is $(x \sqcap a) \sqcup (y \sqcap a)$ is non-least what is equivalent to $x \sqcap a$ or $y \sqcap a$ being non-least that is $x \in \star a \vee y \in \star a$. □

Theorem 496. If $\mathfrak{A}$ is a starrish join-semilattice lattice then

$$\operatorname{atoms}(a \sqcup b) = \operatorname{atoms} a \cup \operatorname{atoms} b$$

for every $a, b \in \mathfrak{A}$.

Proof. For every atom c we have:

$$\begin{aligned} c \in \operatorname{atoms}(a \sqcup b) &\Leftrightarrow \\ c \not\leq a \sqcup b &\Leftrightarrow \\ a \sqcup b \in \star c &\Leftrightarrow \\ a \in \star c \vee b \in \star c &\Leftrightarrow \\ c \not\leq a \vee c \not\leq b &\Leftrightarrow \\ c \in \operatorname{atoms} a \vee c \in \operatorname{atoms} b. \end{aligned}$$

□

5.6.9.1. *Completely starrish posets.*

Definition 497. I will call a poset *completely starrish* when the full star $\star a$ is a principal free star for every element a of this poset.

Obvious 498. Every completely starrish poset is starrish.

Proposition 499. Every complete join infinite distributive lattice is completely starrish.

PROOF. Let $\mathfrak{A}$ be a join infinite distributive lattice, $a \in \mathfrak{A}$. Obviously $\bot \notin \star a$ (if $\bot$ exists); obviously $\star a$ is an upper set. If $\bigsqcup T \in \star a$, then $(\bigsqcup T) \sqcap a$ is non-least that is $\bigsqcup \langle a \sqcap \rangle^* T$ is non-least what is equivalent to $a \sqcap x$ being non-least for some $x \in T$ that is $x \in \star a$. □

THEOREM 500. If $\mathfrak{A}$ is a completely starrish complete lattice lattice then

$$\operatorname{atoms} \bigsqcup T = \bigcup \langle \operatorname{atoms} \rangle^* T.$$

for every $T \in \mathscr{P}\mathfrak{A}$.

PROOF. For every atom c we have:

$$c \in \operatorname{atoms} \bigsqcup T \Leftrightarrow c \not\leq \bigsqcup T \Leftrightarrow \bigsqcup T \in \star c \Leftrightarrow \exists X \in T : X \in \star c \Leftrightarrow$$
$$\exists X \in T : X \not\leq c \Leftrightarrow \exists X \in T : c \in \operatorname{atoms} X \Leftrightarrow c \in \bigcup \langle \operatorname{atoms} \rangle^* T.$$

□

5.7. Basic properties of filters

PROPOSITION 501. $\operatorname{up} \mathcal{A} = \mathcal{A}$ for every filter $\mathcal{A}$ (provided that we equate elements of the base poset $\mathfrak{Z}$ with corresponding principal filters.

PROOF. $A \in \operatorname{up} \mathcal{A} \Leftrightarrow A \sqsupseteq \mathcal{A} \Leftrightarrow \uparrow A \sqsupseteq \mathcal{A} \Leftrightarrow \uparrow A \subseteq \mathcal{A} \Leftrightarrow A \in \mathcal{A}$. □

5.7.1. Minimal and maximal filters.

OBVIOUS 502. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. $\bot^{\mathfrak{A}}$ (equal to the principal filter for the least element of $\mathfrak{Z}$ if it exists) defined by the formula $\operatorname{up} \bot^{\mathfrak{A}} = \mathfrak{Z}$ is the least element of $\mathfrak{A}$.

PROPOSITION 503. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator with greatest element.
3°. $\top^{\mathfrak{A}}$ defined by the formula $\operatorname{up} \top^{\mathfrak{A}} = \{\top^{\mathfrak{Z}}\}$ is the greatest element of $\mathfrak{A}$.

PROOF. Take into account that filters are nonempty. □

5.7.2. Alignment.

DEFINITION 504. I call *down-aligned* filtrator such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $\mathfrak{A}$ and $\mathfrak{Z}$ have common least element. (Let's denote it $\bot$.)

DEFINITION 505. I call *up-aligned* filtrator such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $\mathfrak{A}$ and $\mathfrak{Z}$ have common greatest element. (Let's denote it $\top$.)

OBVIOUS 506.

1°. If $\mathfrak{Z}$ has least element, the primary filtrator is down-aligned.
2°. If $\mathfrak{Z}$ has greatest element, the primary filtrator is up-aligned.

COROLLARY 507. Every powerset filtrator is both up and down-aligned.

We can also define (without requirement of having least and greatest elements, but coinciding with the above definitions if least/greatest elements are present):

DEFINITION 508. I call *weakly down-aligned* filtrator such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that whenever $\bot^{\mathfrak{Z}}$ exists, $\bot^{\mathfrak{A}}$ also exists and $\bot^{\mathfrak{Z}} = \bot^{\mathfrak{A}}$.

DEFINITION 509. I call *weakly up-aligned* filtrator such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that whenever $\top^{\mathfrak{Z}}$ exists, $\top^{\mathfrak{A}}$ also exists and $\top^{\mathfrak{Z}} = \top^{\mathfrak{A}}$.

Obvious 510.

1°. Every up-aligned filtrator is weakly up-aligned.
2°. Every down-aligned filtrator is weakly down-aligned.

Obvious 511.

1°. Every primary filtrator is weakly down-aligned.
2°. Every primary filtrator is weakly up-aligned.

5.8. More advanced properties of filters

5.8.1. Formulas for Meets and Joins of Filters.

Lemma 512. If f is an order embedding from a poset $\mathfrak{A}$ to a complete lattice $\mathfrak{B}$ and $S \in \mathscr{P}\mathfrak{A}$ and there exists such $\mathcal{F} \in \mathfrak{A}$ that $f\mathcal{F} = \bigsqcup^{\mathfrak{B}} \langle f \rangle^* S$, then $\bigsqcup^{\mathfrak{A}} S$ exists and $f \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} \langle f \rangle^* S$.

Proof. f is an order isomorphism from $\mathfrak{A}$ to $\mathfrak{B}|_{\langle f \rangle^* \mathfrak{A}}$. $f\mathcal{F} \in \mathfrak{B}|_{\langle f \rangle^* \mathfrak{A}}$.
Consequently, $\bigsqcup^{\mathfrak{B}} \langle f \rangle^* S \in \mathfrak{B}|_{\langle f \rangle^* \mathfrak{A}}$ and $\bigsqcup^{\mathfrak{B}|_{\langle f \rangle^* \mathfrak{A}}} \langle f \rangle^* S = \bigsqcup^{\mathfrak{B}} \langle f \rangle^* S$.
$f \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}|_{\langle f \rangle^* \mathfrak{A}}} \langle f \rangle^* S$ because f is an order isomorphism.
Combining, $f \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} \langle f \rangle^* S$. □

Corollary 513. If $\mathfrak{B}$ is a complete lattice and $\mathfrak{A}$ is its subset and $S \in \mathscr{P}\mathfrak{A}$ and $\bigsqcup^{\mathfrak{B}} S \in \mathfrak{A}$, then $\bigsqcup^{\mathfrak{A}} S$ exists and $\bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} S$.

Exercise 514. The below theorem does not work for $S = \emptyset$. Formulate the general case.

Theorem 515.

1°. If $\mathfrak{Z}$ is a meet-semilattice, then $\bigsqcup^{\mathfrak{F}(\mathfrak{Z})} S$ exists and $\bigsqcup^{\mathfrak{F}(\mathfrak{Z})} S = \bigcap S$ for every bounded above set $S \in \mathscr{P}\mathfrak{F}(\mathfrak{Z}) \setminus \{\emptyset\}$.
2°. If $\mathfrak{Z}$ is a join-semilattice, then $\bigsqcap^{\mathfrak{I}(\mathfrak{Z})} S$ exists and $\bigsqcap^{\mathfrak{I}(\mathfrak{Z})} S = \bigcap S$ for for every bounded below set $S \in \mathscr{P}\mathfrak{I}(\mathfrak{Z}) \setminus \{\emptyset\}$.

Proof.

1°. Taking into account the lemma, it is enough to prove that $\bigcap S$ is a filter. Let's prove that $\bigcap S$ is nonempty. There is an upper bound $\mathcal{T}$ of S. Take arbitrary $T \in \mathcal{T}$. We have $T \in \mathcal{X}$ for every $\mathcal{X} \in S$. Thus S is nonempty.
For every $A, B \in \mathfrak{Z}$ we have:

$$A, B \in \bigcap S \Leftrightarrow \forall P \in S : A, B \in P \Leftrightarrow \forall P \in S : A \sqcap B \in P \Leftrightarrow A \sqcap B \in \bigcap S.$$

So $\bigcap S$ is a filter.
2°. By duality.

□

Theorem 516.

1°. If $\mathfrak{Z}$ is a meet-semilattice with greatest element, then $\bigsqcup^{\mathfrak{F}(\mathfrak{Z})} S$ exists and $\bigsqcup^{\mathfrak{F}(\mathfrak{Z})} S = \bigcap S$ for every $S \in \mathscr{P}\mathfrak{F}(\mathfrak{Z}) \setminus \{\emptyset\}$.
2°. If $\mathfrak{Z}$ is a join-semilattice with least element, then $\bigsqcap^{\mathfrak{I}(\mathfrak{Z})} S$ exists and $\bigsqcap^{\mathfrak{I}(\mathfrak{Z})} S = \bigcap S$ for every $S \in \mathscr{P}\mathfrak{I}(\mathfrak{Z}) \setminus \{\emptyset\}$.
3°. If $\mathfrak{Z}$ is a join-semilattice with least element, then $\bigsqcup^{\mathfrak{S}(\mathfrak{Z})} S$ exists and $\bigsqcup^{\mathfrak{S}(\mathfrak{Z})} S = \bigcup S$ for every $S \in \mathscr{P}\mathfrak{S}(\mathfrak{Z})$.
4°. If $\mathfrak{Z}$ is a meet-semilattice with greatest element, then $\bigsqcap^{\mathfrak{M}(\mathfrak{Z})} S$ exists and $\bigsqcap^{\mathfrak{M}(\mathfrak{Z})} S = \bigcup S$ for every $S \in \mathscr{P}\mathfrak{M}(\mathfrak{Z})$.

Proof.

1°. From the previous theorem.

2°. By duality.

3°. Taking into account the lemma, it is enough to prove that $\bigcup S$ is a free star. $\bigcup S$ is not the complement of empty set because $\bot \notin \bigcup S$. For every $A, B \in \mathfrak{Z}$ we have:

$$A \in \bigcup S \vee B \in \bigcup S \Leftrightarrow \exists P \in S : (A \in P \vee B \in P) \Leftrightarrow$$
$$\exists P \in S : A \sqcup B \in P \Leftrightarrow A \sqcup B \in \bigcup S.$$

4°. By duality.

□

Corollary 517. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with greatest element $\top$.

3°. $\bigsqcup^{\mathfrak{A}} S$ exists and $\operatorname{up} \bigsqcup^{\mathfrak{A}} S = \bigcap \langle \operatorname{up} \rangle^* S$ for every $S \in \mathscr{P}\mathfrak{A} \setminus \{\emptyset\}$.

Proof.

1°⇒2°. Obvious.

2°⇒3°. By the theorem.

□

Corollary 518. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with greatest element $\top$.

3°. $\mathfrak{A}$ is a complete lattice.

We will denote meets and joins on the lattice of filters just as $\sqcap$ and $\sqcup$.

Proposition 519. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over an ideal base.

3°. $\mathfrak{A}$ is a join-semilattice and for any $\mathcal{A}, \mathcal{B} \in \mathfrak{A}$

$$\operatorname{up}(\mathcal{A} \sqcup^{\mathfrak{A}} \mathcal{B}) = \operatorname{up} \mathcal{A} \cap \operatorname{up} \mathcal{B}.$$

Proof.

1°⇒2°. Obvious.

2°⇒3°. Taking in account the lemma it is enough to prove that $R = \operatorname{up} \mathcal{A} \cap \operatorname{up} \mathcal{B}$ is a filter.

R is nonempty because we can take $X \in \operatorname{up} \mathcal{A}$ and $Y \in \operatorname{up} \mathcal{B}$ and $Z \sqsupseteq X \wedge Z \sqsupseteq Y$ and then $R \ni Z$.

Let $A, B \in R$. Then $A, B \in \operatorname{up} \mathcal{A}$; so exists $C \in \operatorname{up} \mathcal{A}$ such that $C \sqsubseteq A \wedge C \sqsubseteq B$. Analogously exists $D \in \operatorname{up} \mathcal{B}$ such that $D \sqsubseteq A \wedge D \sqsubseteq B$. Take $E \sqsupseteq C \wedge E \sqsupseteq D$. Then $E \in \operatorname{up} \mathcal{A}$ and $E \in \operatorname{up} \mathcal{B}$; $E \in R$ and $E \sqsubseteq A \wedge E \sqsubseteq B$. So R is a filter base.

That R is an upper set is obvious.

□

Theorem 520. Let $\mathfrak{Z}$ be a distributive lattice. Then

1°. $\bigsqcap^{\mathfrak{F}(\mathfrak{Z})} S = \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup S \text{ where } i=0,\dots,n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathscr{P}\mathfrak{F}(\mathfrak{Z}) \setminus \{\emptyset\}$;

2°. $\bigsqcup^{\mathfrak{I}(\mathfrak{Z})} S = \left\{ \frac{K_0 \sqcup^{\mathfrak{Z}} \dots \sqcup^{\mathfrak{Z}} K_n}{K_i \in \bigcup S \text{ where } i=0,\dots,n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathscr{P}\mathfrak{I}(\mathfrak{Z}) \setminus \{\emptyset\}$.

PROOF. We will prove only the first, as the second is dual.

Let's denote the right part of the equality to be proven as R. First we will prove that R is a filter. R is nonempty because S is nonempty.

Let $A, B \in R$. Then $A = X_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} X_k$, $B = Y_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} Y_l$ where $X_i, Y_j \in \bigcup S$. So

$$A \sqcap^{\mathfrak{Z}} B = X_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} X_k \sqcap^{\mathfrak{Z}} Y_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} Y_l \in R.$$

Let element $C \sqsupseteq A \in R$. Consequently (distributivity used)

$$C = C \sqcup^{\mathfrak{Z}} A = (C \sqcup^{\mathfrak{Z}} X_0) \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} (C \sqcup^{\mathfrak{Z}} X_k).$$

$X_i \in P_i$ for some $P_i \in S$; $C \sqcup^{\mathfrak{Z}} X_i \in P_i$; $C \sqcup^{\mathfrak{Z}} X_i \in \bigcup S$; consequently $C \in R$.

We have proved that that R is a filter base and an upper set. So R is a filter.

Let $\mathcal{A} \in S$. Then $\mathcal{A} \subseteq \bigcup S$;

$$R \supseteq \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \mathcal{A} \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} = \mathcal{A}.$$

Consequently $\mathcal{A} \sqsupseteq R$.

Let now $\mathcal{B} \in \mathfrak{A}$ and $\forall \mathcal{A} \in S : \mathcal{A} \sqsupseteq \mathcal{B}$. Then $\forall \mathcal{A} \in S : \mathcal{A} \subseteq \mathcal{B}$; $\mathcal{B} \supseteq \bigcup S$. Thus $\mathcal{B} \supseteq T$ for every finite set $T \subseteq \bigcup S$. Consequently $\operatorname{up} \mathcal{B} \ni \prod^{\mathfrak{Z}} T$. Thus $\mathcal{B} \supseteq R$; $\mathcal{B} \sqsubseteq R$.

Comparing we get $\prod^{\mathfrak{F}(\mathfrak{Z})} S = R$.

□

COROLLARY 521. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice.

3°. $\operatorname{up} \prod^{\mathfrak{A}} S = \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup \langle \operatorname{up} \rangle^* S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathscr{P}\mathfrak{A} \setminus \{\emptyset\}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. By the theorem.

□

THEOREM 522. Let $\mathfrak{Z}$ be a distributive lattice. Then:

1°. $\mathcal{F}_0 \sqcap^{\mathfrak{F}(\mathfrak{Z})} \dots \sqcap^{\mathfrak{F}(\mathfrak{Z})} \mathcal{F}_m = \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_m}{K_i \in \mathcal{F}_i \text{ where } i = 0, \dots, m} \right\}$ for any $\mathcal{F}_0, \dots, \mathcal{F}_m \in \mathfrak{F}(\mathfrak{Z})$;

2°. $\mathcal{F}_0 \sqcup^{\mathfrak{I}(\mathfrak{Z})} \dots \sqcup^{\mathfrak{I}(\mathfrak{Z})} \mathcal{F}_m = \left\{ \frac{K_0 \sqcup^{\mathfrak{Z}} \dots \sqcup^{\mathfrak{Z}} K_m}{K_i \in \mathcal{F}_i \text{ where } i = 0, \dots, m} \right\}$ for any $\mathcal{F}_0, \dots, \mathcal{F}_m \in \mathfrak{I}(\mathfrak{Z})$.

PROOF. We will prove only the first as the second is dual.

Let's denote the right part of the equality to be proven as R. First we will prove that R is a filter. Obviously R is nonempty.

Let $A, B \in R$. Then $A = X_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} X_m$, $B = Y_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} Y_m$ where $X_i, Y_i \in \mathcal{F}_i$.

$$A \sqcap^{\mathfrak{Z}} B = (X_0 \sqcap^{\mathfrak{Z}} Y_0) \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} (X_m \sqcap^{\mathfrak{Z}} Y_m),$$

consequently $A \sqcap^{\mathfrak{Z}} B \in R$.

Let filter $C \sqsupseteq A \in R$

$$C = A \sqcup^{\mathfrak{Z}} C = (X_0 \sqcup^{\mathfrak{Z}} C) \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} (X_m \sqcup^{\mathfrak{Z}} C) \in R.$$

So R is a filter.

Let $P_i \in \mathcal{F}_i$. Then $P_i \in R$ because $P_i = (P_i \sqcup^{\mathfrak{Z}} P_0) \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} (P_i \sqcup^{\mathfrak{Z}} P_m)$. So $\mathcal{F}_i \subseteq R$; $\mathcal{F}_i \sqsupseteq R$.

Let now $\mathcal{B} \in \mathfrak{A}$ and $\forall i \in \{0, \dots, m\} : \mathcal{F}_i \sqsupseteq \mathcal{B}$. Then $\forall i \in \{0, \dots, m\} : \mathcal{F}_i \subseteq \mathcal{B}$.

Let $L_i \in \mathcal{B}$ for every $L_i \in \mathcal{F}_i$. $L_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} L_m \in \mathcal{B}$. So $\mathcal{B} \supseteq R$; $\mathcal{B} \sqsubseteq R$.

So $\mathcal{F}_0 \sqcap^{\mathfrak{F}(\mathfrak{Z})} \dots \sqcap^{\mathfrak{F}(\mathfrak{Z})} \mathcal{F}_m = R$. □

COROLLARY 523. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice.
3°. $\operatorname{up}(\mathcal{F}_0 \sqcap^{\mathfrak{A}} \dots \sqcap^{\mathfrak{A}} \mathcal{F}_m) = \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_m}{K_i \in \operatorname{up} \mathcal{F}_i \text{ where } i=0,\dots,m} \right\}$ for any $\mathcal{F}_0, \dots, \mathcal{F}_m \in \mathfrak{A}$.

PROOF.
1°⇒2°. Obvious.
2°⇒3°. By the theorem.

□

More general case of semilattices follows:

THEOREM 524.

1°. $\prod^{\mathfrak{F}(\mathfrak{Z})} S = \bigcup \left\{ \frac{\uparrow(K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_n)}{K_i \in \bigcup S \text{ where } i=0,\dots,n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathscr{P}\mathfrak{F}(\mathfrak{Z}) \setminus \{\emptyset\}$ if $\mathfrak{Z}$ is a meet-semilattice;
2°. $\bigsqcup^{\mathfrak{I}(\mathfrak{Z})} S = \bigcup \left\{ \frac{\uparrow(K_0 \sqcup^{\mathfrak{Z}} \dots \sqcup^{\mathfrak{Z}} K_n)}{K_i \in \bigcup S \text{ where } i=0,\dots,n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathscr{P}\mathfrak{I}(\mathfrak{Z}) \setminus \{\emptyset\}$ if $\mathfrak{Z}$ is a join-semilattice.

PROOF. We will prove only the first as the second is dual.
It follows from the fact that

$$\prod^{\mathfrak{F}(\mathfrak{Z})} S = \prod^{\mathfrak{F}(\mathfrak{Z})} \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\}$$

and that $\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup S \text{ where } i=0,\dots,n \text{ for } n \in \mathbb{N}} \right\}$ is a filter base. □

COROLLARY 525. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice.
3°. $\operatorname{up} \prod S = \bigcup \left\{ \frac{\operatorname{up}(K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_n)}{K_i \in \bigcup \langle \operatorname{up} \rangle^* S \text{ where } i=0,\dots,n \text{ for } n \in \mathbb{N}} \right\}$ for every $S \in \mathscr{P}\mathfrak{A} \setminus \{\emptyset\}$.

THEOREM 526.

1°. $\mathcal{F}_0 \sqcap^{\mathfrak{F}(\mathfrak{Z})} \dots \sqcap^{\mathfrak{F}(\mathfrak{Z})} \mathcal{F}_m = \bigcup \left\{ \frac{\uparrow(K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_m)}{K_i \in \mathcal{F}_i \text{ where } i=0,\dots,m} \right\}$ for $S \in \mathscr{P}\mathfrak{F}(\mathfrak{Z}) \setminus \{\emptyset\}$ if $\mathfrak{Z}$ is a meet-semilattice;
2°. $\mathcal{F}_0 \sqcup^{\mathfrak{I}(\mathfrak{Z})} \dots \sqcup^{\mathfrak{I}(\mathfrak{Z})} \mathcal{F}_m = \bigcup \left\{ \frac{\uparrow(K_0 \sqcup^{\mathfrak{Z}} \dots \sqcup^{\mathfrak{Z}} K_m)}{K_i \in \mathcal{F}_i \text{ where } i=0,\dots,m} \right\}$ for $S \in \mathscr{P}\mathfrak{I}(\mathfrak{Z}) \setminus \{\emptyset\}$ if $\mathfrak{Z}$ is a join-semilattice.

PROOF. We will prove only the first as the second is dual.
It follows from the fact that

$$\mathcal{F}_0 \sqcap^{\mathfrak{F}(\mathfrak{Z})} \dots \sqcap^{\mathfrak{F}(\mathfrak{Z})} \mathcal{F}_m = \prod^{\mathfrak{F}(\mathfrak{Z})} \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_m}{K_i \in \mathcal{F}_i \text{ where } i = 0, \dots, m} \right\}$$

and that $\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_m}{K_i \in \mathcal{F}_i \text{ where } i=0,\dots,m} \right\}$ is a filter base. □

COROLLARY 527. $\operatorname{up}(\mathcal{F}_0 \sqcap^{\mathfrak{F}(\mathfrak{Z})} \dots \sqcap^{\mathfrak{F}(\mathfrak{Z})} \mathcal{F}_m) = \bigcup \left\{ \frac{\operatorname{up}(K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_m)}{K_i \in \mathcal{F}_i \text{ where } i=0,\dots,m} \right\}$ if $\mathfrak{Z}$ is a meet-semilattice.

LEMMA 528. If $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator and $\mathfrak{Z}$ is a meet-semilattice and an ideal base, then $\mathfrak{A}$ is a lattice.

PROOF. It is a join-semilattice by proposition 519. It is a meet-semilattice by theorem 524. □

Corollary 529. If $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator and $\mathfrak{Z}$ is a lattice, then $\mathfrak{A}$ is a lattice.

5.8.2. Distributivity of the Lattice of Filters.

Theorem 530. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice.
3°. $\mathcal{A} \sqcup^{\mathfrak{A}} \bigsqcap^{\mathfrak{A}} S = \bigsqcap^{\mathfrak{A}} \langle \mathcal{A} \sqcup^{\mathfrak{A}} \rangle^* S$ for $S \in \mathscr{P}\mathfrak{A}$ and $\mathcal{A} \in \mathfrak{A}$.

Proof.

1°⇒2°. Obvious.

2°⇒3°. Taking into account the previous section, we have:

$$\begin{aligned}
\operatorname{up}\left(\mathcal{A} \sqcup^{\mathfrak{A}} \bigsqcap^{\mathfrak{A}} S\right) &= \\
\operatorname{up}\mathcal{A} \cap \operatorname{up}\bigsqcap^{\mathfrak{A}} S &= \\
\operatorname{up}\mathcal{A} \cap \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup \langle \operatorname{up} \rangle^* S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} &= \\
\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n \in \operatorname{up}\mathcal{A}, K_i \in \bigcup \langle \operatorname{up} \rangle^* S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} &= \\
\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \operatorname{up}\mathcal{A}, K_i \in \bigcup \langle \operatorname{up} \rangle^* S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} &= \\
\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \operatorname{up}\mathcal{A} \cap \bigcup \langle \operatorname{up} \rangle^* S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} &= \\
\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup \langle \operatorname{up}\mathcal{A} \cap \rangle^* \langle \operatorname{up} \rangle^* S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} &= \\
\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup \left\{ \frac{\operatorname{up}\mathcal{A} \cap \operatorname{up}\mathcal{X}}{\mathcal{X} \in S} \right\} \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} &= \\
\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup \left\{ \frac{\operatorname{up}(\mathcal{A} \sqcup^{\mathfrak{A}} \mathcal{X})}{\mathcal{X} \in S} \right\} \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} &= \\
\operatorname{up}\bigsqcap^{\mathfrak{A}} \left\{ \frac{\mathcal{A} \sqcup^{\mathfrak{A}} \mathcal{X}}{\mathcal{X} \in S} \right\} &= \\
\operatorname{up}\bigsqcap^{\mathfrak{A}} \langle \mathcal{A} \sqcup^{\mathfrak{A}} \rangle^* S. &
\end{aligned}$$

□

Corollary 531. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice which is an ideal base.
3°. $\mathfrak{A}$ is a distributive and co-brouwerian lattice.

Corollary 532. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice with greatest element.

3°. $\mathfrak{A}$ is a co-frame.

The below theorem uses the notation and results from section 3.9.

THEOREM 533. If $\mathfrak{A}$ is a co-frame and L is a bounded distributive lattice which, then $\operatorname{Join}(L, \mathfrak{A})$ is also a co-frame.

PROOF. Let $F = \uparrow \circ \bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \operatorname{Up}(\mathfrak{A})$; F is a co-nucleus by above.

Since $\operatorname{Up}(\mathfrak{A}) \cong \mathbf{Pos}(\mathfrak{A}, 2)$ by proposition 340, we may regard F as a co-nucleus on $\mathbf{Pos}(\mathfrak{A}, 2)$.

$\operatorname{Join}(L, \mathfrak{A}) \cong \operatorname{Join}(L, \operatorname{Fix}(F))$ by corollary 343.

$\operatorname{Join}(L, \operatorname{Fix}(F)) \cong \operatorname{Fix}(\operatorname{Join}(L, F))$ by lemma 351.

By corollary 350 the function $\operatorname{Join}(L, F)$ is a co-nucleus on $\operatorname{Join}(L, \mathbf{Pos}(\mathfrak{A}, 2))$.

$$\begin{aligned} \operatorname{Join}(L, \mathbf{Pos}(\mathfrak{A}, 2)) &\cong \quad \text{(by lemma 353)} \\ \mathbf{Pos}(\mathfrak{A}, \operatorname{Join}(L, 2)) &\cong \\ \mathbf{Pos}(\mathfrak{A}, \mathfrak{F}(X)). \end{aligned}$$

$\mathfrak{F}(X)$ is a co-frame by corollary 532. Thus $\mathbf{Pos}(\mathfrak{A}, \mathfrak{F}(X))$ is a co-frame by lemma 353.

Thus $\operatorname{Join}(L, \mathfrak{A})$ is isomorphic to a poset of fixed points of a co-nucleus on the co-frame $\mathbf{Pos}(\mathfrak{A}, \mathfrak{F}(X))$. By lemma 335 $\operatorname{Join}(L, \mathfrak{A})$ is also a co-frame. □

5.9. Misc filtrator properties

THEOREM 534. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered filtrator.
4°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with join-closed core.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. The formula $\forall a, b \in \mathfrak{A} : (\operatorname{up} a \supseteq \operatorname{up} b \Rightarrow a \sqsubseteq b)$ is obvious for primary filtrators.

3°⇒4°. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtered filtrator. Let $S \in \mathscr{P}\mathfrak{Z}$ and $\bigsqcup^{\mathfrak{Z}} S$ be defined. We need to prove $\bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{Z}} S$. That $\bigsqcup^{\mathfrak{Z}} S$ is an upper bound for S is obvious. Let $a \in \mathfrak{A}$ be an upper bound for S. It's enough to prove that $\bigsqcup^{\mathfrak{Z}} S \sqsubseteq a$. Really,

$$c \in \operatorname{up} a \Rightarrow c \sqsupseteq a \Rightarrow \forall x \in S : c \sqsupseteq x \Rightarrow c \sqsupseteq \overset{\mathfrak{Z}}{\bigsqcup} S \Rightarrow c \in \operatorname{up} \overset{\mathfrak{Z}}{\bigsqcup} S;$$

so $\operatorname{up} a \subseteq \operatorname{up} \bigsqcup^{\mathfrak{Z}} S$ and thus $a \sqsupseteq \bigsqcup^{\mathfrak{Z}} S$ because it is filtered.

□

5.10. Characterization of Binarily Meet-Closed Filtrators

THEOREM 535. The following are equivalent for a filtrator $(\mathfrak{A}, \mathfrak{Z})$ whose core is a meet semilattice such that $\forall a \in \mathfrak{A} : \operatorname{up} a \neq \emptyset$:

1°. The filtrator is with binarily meet-closed core.
2°. $\operatorname{up} a$ is a filter for every $a \in \mathfrak{A}$.

PROOF.

1°⇒2°. Let $X, Y \in \operatorname{up} a$. Then $X \sqcap^{\mathfrak{Z}} Y = X \sqcap^{\mathfrak{A}} Y \sqsupseteq a$. That $\operatorname{up} a$ is an upper set is obvious. So taking into account that $\operatorname{up} a \neq \emptyset$, $\operatorname{up} a$ is a filter.

2°⇒1°. It is enough to prove that $a \sqsubseteq A, B \Rightarrow a \sqsubseteq A \sqcap^{\mathfrak{Z}} B$ for every $A, B \in \mathfrak{A}$. Really:

$$a \sqsubseteq A, B \Rightarrow A, B \in \operatorname{up} a \Rightarrow A \sqcap^{\mathfrak{Z}} B \in \operatorname{up} a \Rightarrow a \sqsubseteq A \sqcap^{\mathfrak{Z}} B.$$

□

COROLLARY 536. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet semilattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is with binarily meet-closed core.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. From the theorem.

□

5.10.1. Separability of Core for Primary Filtrators.

THEOREM 537. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet semilattice with least element.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is with separable core.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Let $\mathcal{A} \asymp^{\mathfrak{A}} \mathcal{B}$ where $\mathcal{A}, \mathcal{B} \in \mathfrak{A}$.

$$\operatorname{up}(\mathcal{A} \sqcap^{\mathfrak{A}} \mathcal{B}) = \bigcup \left\{ \frac{\operatorname{up}(A \sqcap^{\mathfrak{Z}} B)}{A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}} \right\}.$$

So

$$\begin{aligned} \bot \in \operatorname{up}(\mathcal{A} \sqcap^{\mathfrak{A}} \mathcal{B}) &\Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B} : \bot \in \operatorname{up}(A \sqcap^{\mathfrak{Z}} B) &\Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B} : A \sqcap^{\mathfrak{Z}} B = \bot &\Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B} : A \sqcap^{\mathfrak{A}} B = \bot^{\mathfrak{A}} & \end{aligned}$$

(used proposition 536).

□

5.11. Core Part

Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator.

DEFINITION 538. The *core part* of an element $a \in \mathfrak{A}$ is $\operatorname{Cor} a = \bigsqcap^{\mathfrak{Z}} \operatorname{up} a$.

DEFINITION 539. The *dual core part* of an element $a \in \mathfrak{A}$ is $\operatorname{Cor}' a = \bigsqcup^{\mathfrak{Z}} \operatorname{down} a$.

OBVIOUS 540. Cor' is dual of Cor.

OBVIOUS 541. $\operatorname{Cor} a = \operatorname{Cor}' a = a$ for every element a of the core of a filtrator.

THEOREM 542. The following is an implications tuple:

1°. a is a filter on a set.
2°. a is a filter on a complete lattice.
3°. a is an element of a filtered filtrator and $\operatorname{Cor} a$ exists.
4°. $\operatorname{Cor} a \sqsubseteq a$ and $\operatorname{Cor} a \in \operatorname{down} a$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Theorem 534.
3°⇒4°. $\operatorname{Cor} a = \bigsqcap^{\mathfrak{Z}} \operatorname{up} a \sqsubseteq \bigsqcap^{\mathfrak{A}} \operatorname{up} a = a$. Then obviously $\operatorname{Cor} a \in \operatorname{down} a$.

□

THEOREM 543. The following is an implications tuple:

1°. a is a filter on a set.
2°. a is a filter on a complete lattice.
3°. a is an element a of a filtrator with join-closed core and $\operatorname{Cor}' a$ exists.
4°. $\operatorname{Cor}' a \sqsubseteq a$ and $\operatorname{Cor}' a \in \operatorname{down} a$ and $\operatorname{Cor}' a = \max \operatorname{down} a$.

PROOF.
1°⇒2°. Obvious.
2°⇒3°. It is join closed by 534. $\operatorname{Cor}' a$ exists because our filtrator is join-closed.
3°⇒4°. $\operatorname{Cor}' a = \bigsqcup^{\mathfrak{Z}} \operatorname{down} a = \bigsqcup^{\mathfrak{A}} \operatorname{down} a \sqsubseteq a$. Now $\operatorname{Cor}' a \in \operatorname{down} a$ is obvious. Thus $\operatorname{Cor}' a = \max \operatorname{down} a$.

□

PROPOSITION 544. $\operatorname{Cor}' a \sqsubseteq \operatorname{Cor} a$ whenever both $\operatorname{Cor} a$ and $\operatorname{Cor}' a$ exist for any element a of a filtrator with join-closed core.

PROOF. $\operatorname{Cor} a = \bigsqcap^{\mathfrak{Z}} \operatorname{up} a \sqsupseteq \operatorname{Cor}' a$ because $\forall A \in \operatorname{up} a : \operatorname{Cor}' a \sqsubseteq A$. □

THEOREM 545. The following is an implications tuple:

1°. a is a filter on a set.
2°. a is a filter on a complete lattice.
3°. a is an element of a filtered filtrator and both $\operatorname{Cor} a$ and $\operatorname{Cor}' a$ exist.
4°. $\operatorname{Cor}' a = \operatorname{Cor} a$.

PROOF.
1°⇒2°. Obvious.
2°⇒3°. By theorem 534.
3°⇒4°. It is with join-closed core because it is filtered. So $\operatorname{Cor}' a \sqsubseteq \operatorname{Cor} a$. $\operatorname{Cor} a \in \operatorname{down} a$. So $\operatorname{Cor} a \sqsubseteq \bigsqcup^{\mathfrak{Z}} \operatorname{down} a = \operatorname{Cor}' a$.

□

COROLLARY 546. $\operatorname{Cor}' a = \operatorname{Cor} a = \bigcap a$ for every filter a on a set.

5.12. Intersection and Joining with an Element of the Core

DEFINITION 547. A filtrator $(\mathfrak{A}; \mathfrak{Z})$ is with *correct intersection* iff $\forall a, b \in \mathfrak{Z} : (a \not\asymp^{\mathfrak{Z}} b \Leftrightarrow a \not\asymp^{\mathfrak{A}} b)$.

DEFINITION 548. A filtrator $(\mathfrak{A}; \mathfrak{Z})$ is with *correct joining* iff $\forall a, b \in \mathfrak{Z} : (a \equiv^{\mathfrak{Z}} b \Leftrightarrow a \equiv^{\mathfrak{A}} b)$.

PROPOSITION 549. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is with binarily meet-closed core, weakly down-aligned filtrator, and $\mathfrak{Z}$ is a meet-semilattice.
4°. $(\mathfrak{A}, \mathfrak{Z})$ is with correct intersection.

PROOF.
1°⇒2°. Obvious.
2°⇒3°. Corollary 536.

3°⇒4°. $a \not\asymp^{\mathfrak{Z}} b \Rightarrow a \not\asymp^{\mathfrak{A}} b$ is obvious. Let $a \asymp^{\mathfrak{Z}} b$. Then $a \sqcap^{\mathfrak{Z}} b$ exists; so $\bot^{\mathfrak{Z}}$ exists and $a \sqcap^{\mathfrak{Z}} b = \bot^{\mathfrak{Z}}$ (as otherwise $a \sqcap^{\mathfrak{Z}} b$ is non-least). So $\bot^{\mathfrak{Z}} = \bot^{\mathfrak{A}}$. We have $a \sqcap^{\mathfrak{A}} b = \bot^{\mathfrak{A}}$. Thus $a \asymp^{\mathfrak{A}} b$.

□

PROPOSITION 550. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a join-semilattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is with binarily join-closed core, weakly up-aligned filtrator, and $\mathfrak{Z}$ is a join-semilattice.
4°. $(\mathfrak{A}, \mathfrak{Z})$ is with correct joining.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Corollary 534.
3°⇒4°. Dual of the previous proposition.

□

LEMMA 551. For a filtrator $(\mathfrak{A}, \mathfrak{Z})$ where $\mathfrak{Z}$ is a boolean lattice, for every $B \in \mathfrak{Z}$, $\mathcal{A} \in \mathfrak{A}$:

1°. $B \asymp^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \sqsupseteq \mathcal{A}$ if it is with separable core and with correct intersection;
2°. $B \equiv^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \sqsubseteq \mathcal{A}$ if it is with co-separable core and with correct joining.

PROOF. We will prove only the first as the second is dual.

$$\begin{aligned} B \asymp^{\mathfrak{A}} \mathcal{A} &\Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A} : B \asymp^{\mathfrak{A}} A &\Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A} : B \asymp^{\mathfrak{Z}} A &\Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A} : \overline{B} \sqsupseteq A &\Leftrightarrow \\ \overline{B} \in \operatorname{up} \mathcal{A} &\Leftrightarrow \\ \overline{B} \sqsupseteq \mathcal{A}. \end{aligned}$$

□

5.13. Stars of Elements of Filtrators

DEFINITION 552. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. *Core star* of an element a of the filtrator is

$$\partial a = \left\{ \frac{x \in \mathfrak{Z}}{x \not\asymp^{\mathfrak{A}} a} \right\}.$$

PROPOSITION 553. $\operatorname{up} a \subseteq \partial a$ for any non-least element a of a filtrator.

PROOF. For any element $X \in \mathfrak{Z}$

$$X \in \operatorname{up} a \Rightarrow a \sqsubseteq X \wedge a \sqsubseteq a \Rightarrow X \not\asymp^{\mathfrak{A}} a \Rightarrow X \in \partial a.$$

□

THEOREM 554. Let $(\mathfrak{A}, \mathfrak{Z})$ be a distributive lattice filtrator with least element and binarily join-closed core which is a join-semilattice. Then ∂a is a free star for each $a \in \mathfrak{A}$.

PROOF. For every $A, B \in \mathfrak{Z}$

$$\begin{aligned} A \sqcup^{\mathfrak{Z}} B \in \partial a &\Leftrightarrow \\ A \sqcup^{\mathfrak{A}} B \in \partial a &\Leftrightarrow \\ (A \sqcup^{\mathfrak{A}} B) \sqcap^{\mathfrak{A}} a \neq \bot^{\mathfrak{A}} &\Leftrightarrow \\ (A \sqcap^{\mathfrak{A}} a) \sqcup^{\mathfrak{A}} (B \sqcap^{\mathfrak{A}} a) \neq \bot^{\mathfrak{A}} &\Leftrightarrow \\ A \sqcap^{\mathfrak{A}} a \neq \bot^{\mathfrak{A}} \vee B \sqcap^{\mathfrak{A}} a \neq \bot^{\mathfrak{A}} &\Leftrightarrow \\ A \in \partial a \vee B \in \partial a. \end{aligned}$$

That ∂a doesn't contain $\bot^{\mathfrak{A}}$ is obvious. □

DEFINITION 555. I call a filtrator *star-separable* when its core is a separation subset of its base.

5.14. Atomic Elements of a Filtrator

See [4, 9] for more detailed treatment of ultrafilters and prime filters.

PROPOSITION 556. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with greatest element.
3°. $\mathfrak{A}$ is a complete lattice.
4°. $\operatorname{atoms} \bigsqcap S = \bigcap \langle \operatorname{atoms} \rangle^* S$ for every $S \in \mathscr{P}\mathfrak{A}$.
5°. $\operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$ for $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Corollary 518.
3°⇒4°. Theorem 108.
4°⇒5°. Obvious.

□

PROPOSITION 557. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice which is and ideal base.
3°. $\mathfrak{A}$ is a starrish join-semilattice.
4°. $\operatorname{atoms}(a \sqcup b) = \operatorname{atoms} a \cup \operatorname{atoms} b$ for $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Corollary 531.
3°⇒4°. Corollary 496.

□

THEOREM 558. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered weakly down-aligned filtrator with binarily meet-closed core $\mathfrak{Z}$ which is a meet-semilattice.
4°. a is an atom of $\mathfrak{Z}$ iff $a \in \mathfrak{Z}$ and a is an atom of $\mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. It is filtered by the theorem 534, binarily meet-closed by corollary 536.

3°⇒4°.

⇐. Let a be an atom of $\mathfrak{A}$ and $a \in \mathfrak{Z}$. Then either a is an atom of $\mathfrak{Z}$ or a is the least element of $\mathfrak{Z}$. But if a is the least element of $\mathfrak{Z}$ then a is also least element of $\mathfrak{A}$ and thus is not an atom of $\mathfrak{A}$. So the only possible outcome is that a is an atom of $\mathfrak{Z}$.

⇒. We need to prove that if a is an atom of $\mathfrak{Z}$ then a is an atom of $\mathfrak{A}$. Suppose the contrary that a is not an atom of $\mathfrak{A}$. Then there exists $x \in \mathfrak{A}$ such that $x \sqsubset a$ and x is not least element of $\mathfrak{A}$. Because "up" is a straight monotone map to the dual of the poset $\mathscr{P}\mathfrak{Z}$ (obvious 454), $\operatorname{up} a \subset \operatorname{up} x$. So there exists $K \in \operatorname{up} x$ such that $K \notin \operatorname{up} a$. Also $a \in \operatorname{up} x$. We have $K \sqcap^{\mathfrak{Z}} a = K \sqcap^{\mathfrak{A}} a \in \operatorname{up} x$; $K \sqcap^{\mathfrak{Z}} a$ is not least of $\mathfrak{Z}$ (Suppose for the contrary that $K \sqcap^{\mathfrak{Z}} a = \bot^{\mathfrak{Z}}$, then $K \sqcap^{\mathfrak{Z}} a = \bot^{\mathfrak{A}} \notin \operatorname{up} x$.) and $K \sqcap^{\mathfrak{Z}} a \sqsubset a$. So a is not an atom of $\mathfrak{Z}$.

□

THEOREM 559. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered filtrator.
4°. $a \in \mathfrak{A}$ is an atom of $\mathfrak{A}$ iff $\operatorname{up} a = \partial a$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. By the theorem 534.

3°⇒4°.

⇒. For any $K \in \mathfrak{A}$

$$K \in \operatorname{up} a \Leftrightarrow K \sqsupseteq a \Leftrightarrow K \not\asymp^{\mathfrak{A}} a \Leftrightarrow K \in \partial a.$$

⇐. Let $\operatorname{up} a = \partial a$. Then a is not least element of $\mathfrak{A}$. Consequently for every $x \in \mathfrak{A}$ if x is not the least element of $\mathfrak{A}$ we have

$$\begin{aligned} x \sqsubset a &\Rightarrow \\ x \not\asymp^{\mathfrak{A}} a &\Rightarrow \\ \forall K \in \operatorname{up} x : K \in \partial a &\Rightarrow \\ \forall K \in \operatorname{up} x : K \in \operatorname{up} a &\Rightarrow \\ \operatorname{up} x \subseteq \operatorname{up} a &\Rightarrow \\ x \sqsupseteq a. \end{aligned}$$

So a is an atom of $\mathfrak{A}$.

□

PROPOSITION 560. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. Coatoms of $\mathfrak{A}$ are exactly coatoms of $\mathfrak{Z}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. Suppose a is a coatom of $\mathfrak{Z}$. Then a is the only non-greatest element in $\operatorname{up} a$. Suppose $b \sqsupset a$ for some $b \in \mathfrak{A}$. Then a cannot be in $\operatorname{up} b$ and thus the only possible element of $\operatorname{up} b$ is the greatest element of $\mathfrak{Z}$ (if it exists) from what follows $b = \top^{\mathfrak{A}}$. So a is a coatom of $\mathfrak{A}$.

Suppose now that a is a coatom of $\mathfrak{A}$. To finish the proof it is enough to show that a is principal. (Then a is non-greatest and thus is a coatom of $\mathfrak{Z}$.)

Suppose a is non-principal. Then obviously exist two distinct elements x and y of the core such that $x, y \in \operatorname{up} a$. Thus a is not an atom of $\mathfrak{A}$.

□

COROLLARY 561. Coatoms of the set of filters on a set U are exactly sets $U \setminus \{x\}$ where $x \in U$.

PROPOSITION 562. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a coatomic poset.
3°. $\mathfrak{A}$ is coatomic.

PROOF.

1°⇒2°. Obvious.

2°⇒2°. Suppose $\mathcal{A} \in \mathfrak{A}$ and $\mathcal{A} \neq \top^{\mathfrak{A}}$. Then there exists $A \in \operatorname{up} \mathcal{A}$ such that A is not greatest element of $\mathfrak{Z}$. Consequently there exists a coatom $a \in \mathfrak{Z}$ such that $a \sqsupseteq A$. Thus $a \in \operatorname{up} \mathcal{A}$ and a is not greatest.

□

5.15. Prime Filtrator Elements

DEFINITION 563. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. *Prime* filtrator elements are such $a \in \mathfrak{A}$ that $\operatorname{up} a$ is a free star (in lattice $\mathfrak{Z}$).

PROPOSITION 564. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice which is an ideal base.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with binarily join-closed core, where $\mathfrak{A}$ is a starrish join-semilattice and $\mathfrak{Z}$ is a join-semilattice.
4°. Atomic elements of this filtrator are prime.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. $(\mathfrak{A}, \mathfrak{Z})$ is with binarily join-closed core by the theorem 534, $\mathfrak{A}$ is a distributive lattice by theorem 531.

3°⇒4°. Let a be an atom of the lattice $\mathfrak{A}$. We have for every $X, Y \in \mathfrak{Z}$

$$\begin{aligned} X \sqcup^{\mathfrak{Z}} Y \in \operatorname{up} a &\Leftrightarrow \\ X \sqcup^{\mathfrak{A}} Y \in \operatorname{up} a &\Leftrightarrow \\ X \sqcup^{\mathfrak{A}} Y \sqsupseteq a &\Leftrightarrow \\ X \sqcup^{\mathfrak{A}} Y \not\asymp^{\mathfrak{A}} a &\Leftrightarrow \\ X \not\asymp^{\mathfrak{A}} a \vee Y \not\asymp^{\mathfrak{A}} a &\Leftrightarrow \\ X \sqsupseteq a \vee Y \sqsupseteq a &\Leftrightarrow \\ X \in \operatorname{up} a \vee Y \in \operatorname{up} a. \end{aligned}$$

□

The following theorem is essentially borrowed from [19]:

THEOREM 565. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. Let $a \in \mathfrak{A}$. Then the following are equivalent:
(a) a is prime.
(b) For every $A \in \mathfrak{Z}$ exactly one of $\{A, \overline{A}\}$ is in $\operatorname{up} a$.
(c) a is an atom of $\mathfrak{A}$.

PROOF.
1°⇒2°. Obvious.
2°⇒3°.
3°a⇒3°b. Let a be prime. Then $A \sqcup^{\mathfrak{Z}} \overline{A} = \top^{\mathfrak{A}} \in \operatorname{up} a$. Therefore $A \in \operatorname{up} a \vee \overline{A} \in \operatorname{up} a$. But since $A \sqcap^{\mathfrak{Z}} \overline{A} = \bot^{\mathfrak{Z}}$ it is impossible $A \in \operatorname{up} a \wedge \overline{A} \in \operatorname{up} a$.
3°b⇒3°c. Obviously $a \neq \bot^{\mathfrak{A}}$.
Let a filter $b \sqsubset a$. Take $X \in \operatorname{up} b$ such that $X \notin \operatorname{up} a$. Then $\overline{X} \in \operatorname{up} a$ because a is prime and thus $\overline{X} \in \operatorname{up} b$. So $\bot^{\mathfrak{Z}} = X \sqcap^{\mathfrak{Z}} \overline{X} \in \operatorname{up} b$ and thus $b = \bot^{\mathfrak{A}}$. So a is atomic.
3°c⇒3°a. By the previous proposition.

□

5.16. Stars for filters

THEOREM 566. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice which is an ideal base and has least element.
3°. ∂a is a free star for each $a \in \mathfrak{A}$.

PROOF.
1°⇒2°. Obvious.
2°⇒3°. $\mathfrak{A}$ is a distributive lattice by the corollary 531. The filtrator $(\mathfrak{A}, \mathfrak{Z})$ is binarily join-closed by corollary 534. So we can apply the theorem 554.

□

5.16.1. Stars of Filters on Boolean Lattices. In this section we will consider the set of filters $\mathfrak{A}$ on a boolean lattice $\mathfrak{Z}$.

THEOREM 567. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $\partial \mathcal{A} = \neg \langle \neg \rangle^* \operatorname{up} \mathcal{A} = \langle \neg \rangle^* \neg \operatorname{up} \mathcal{A}$ and $\operatorname{up} \mathcal{A} = \neg \langle \neg \rangle^* \partial \mathcal{A} = \langle \neg \rangle^* \neg \partial \mathcal{A}$.

PROOF.
1°⇒2°. Obvious.
2°⇒3°. Because of properties of diagram (1), it is enough to prove just $\partial \mathcal{A} = \neg \langle \neg \rangle^* \operatorname{up} \mathcal{A}$. Really,

$$X \in \operatorname{up} \mathcal{A} \Leftrightarrow X \sqsupseteq \mathcal{A} \Leftrightarrow \overline{X} \asymp^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{X} \notin \partial \mathcal{A}$$

for any $X \in \mathfrak{Z}$ (taking into account theorems 535, 537, and lemma 551).

□

COROLLARY 568. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. ∂ is an order isomorphism from $\mathfrak{A}$ to $\mathfrak{S}(\mathfrak{Z})$.

PROOF. By properties of the diagram (1). □

Corollary 569. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $\partial \bigsqcup^{\mathfrak{A}} S = \bigcup \langle \partial \rangle^* S$ for every $S \in \mathscr{P}\mathfrak{A}$.

Proof.

1°⇒2°. Obvious.
2°⇒3°. $\partial \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{S}(\mathfrak{Z})} \langle \partial \rangle^* S = \bigcup \langle \partial \rangle^* S$.

□

5.17. Generalized Filter Base

Definition 570. *Generalized filter base* is a filter base on the set $\mathfrak{A}$ where $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.

Definition 571. If S is a generalized filter base and $\mathcal{A} = \bigsqcap^{\mathfrak{A}} S$ for some $\mathcal{A} \in \mathfrak{A}$, then we call S a generalized filter base of $\mathcal{A}$.

Theorem 572. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice.
3°. For a generalized filter base S of $\mathcal{F} \in \mathfrak{A}$ and $K \in \mathfrak{Z}$ we have

$$K \in \operatorname{up} \mathcal{F} \Leftrightarrow \exists \mathcal{L} \in S : K \in \operatorname{up} \mathcal{L}.$$

Proof.

1°⇒2°. Obvious.
2°⇒3°.

⇐. Because $\mathcal{F} = \bigsqcap^{\mathfrak{A}} S$.
⇒. Let $K \in \operatorname{up} \mathcal{F}$. Then (taken into account corollary 525 and that S is nonempty) there exist $X_1, \dots, X_n \in \bigcup \langle \operatorname{up} \rangle^* S$ such that $K \in \operatorname{up}(X_1 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} X_n)$ that is $K \in \operatorname{up}(\uparrow X_1 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} \uparrow X_n)$. Consequently (by theorem 535) $K \in \operatorname{up}(\uparrow X_1 \sqcap^{\mathfrak{A}} \dots \sqcap^{\mathfrak{A}} \uparrow X_n)$. Replacing every $\uparrow X_i$ with such $\mathcal{X}_i \in S$ that $X_i \in \operatorname{up} \mathcal{X}_i$ (this is obviously possible to do), we get a finite set $T_0 \subseteq S$ such that $K \in \operatorname{up} \bigsqcap^{\mathfrak{A}} T_0$. From this there exists $\mathcal{C} \in S$ such that $\mathcal{C} \sqsubseteq \bigsqcap^{\mathfrak{A}} T_0$ and so $K \in \operatorname{up} \mathcal{C}$.

□

Corollary 573. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with least element.
3°. For a generalized filter base S of a $\mathcal{F} \in \mathfrak{A}$ we have

$$\bot^{\mathfrak{A}} \in S \Leftrightarrow \mathcal{F} = \bot^{\mathfrak{A}}.$$

Proof. Substitute $\bot^{\mathfrak{A}}$ as K. □

Theorem 574. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with least element.
3°. Let $\mathcal{F}_0 \sqcap^{\mathfrak{A}} \dots \sqcap^{\mathfrak{A}} \mathcal{F}_n \neq \bot^{\mathfrak{A}}$ for every $\mathcal{F}_0, \dots, \mathcal{F}_n \in S$, where S is a nonempty set of elements of $\mathfrak{A}$. Then $\bigsqcap^{\mathfrak{A}} S \neq \bot^{\mathfrak{A}}$.

Proof. Consider the set

$$S' = \left\{ \frac{\mathcal{F}_0 \sqcap^{\mathfrak{A}} \dots \sqcap^{\mathfrak{A}} \mathcal{F}_n}{\mathcal{F}_0, \dots, \mathcal{F}_n \in S} \right\}.$$

Obviously S' is nonempty and binarily meet-closed. So S' is a generalized filter base. Obviously $\bot^{\mathfrak{A}} \notin S$. So by properties of generalized filter bases $\bigsqcap^{\mathfrak{A}} S' \neq \bot^{\mathfrak{A}}$. But obviously $\bigsqcap^{\mathfrak{A}} S = \bigsqcap^{\mathfrak{A}} S'$. So $\bigsqcap^{\mathfrak{A}} S \neq \bot^{\mathfrak{A}}$. □

Corollary 575. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with least element.
3°. Let $S \in \mathscr{P}\mathfrak{Z}$ such that $S \neq \emptyset$ and $A_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} A_n \neq \bot^{\mathfrak{Z}}$ for every $A_0, \dots, A_n \in S$. Then $\bigsqcap^{\mathfrak{A}} S \neq \bot^{\mathfrak{A}}$.

Proof.

1°⇒2°. Obvious.

2°⇒3°. Because $(\mathfrak{A}, \mathfrak{Z})$ is binarily meet-closed (by the theorem 535).

□

Theorem 576. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a bounded meet-semilattice.
3°. $\mathfrak{A}$ is an atomic lattice.

Proof.

1°⇒2°. Obvious.

2°⇒3°. Let $\mathcal{F} \in \mathfrak{A}$. Let choose (by Kuratowski's lemma) a maximal chain S from $\bot^{\mathfrak{A}}$ to $\mathcal{F}$. Let $S' = S \setminus \{\bot^{\mathfrak{A}}\}$. $a = \bigsqcap^{\mathfrak{A}} S' \neq \bot^{\mathfrak{A}}$ by properties of generalized filter bases (the corollary 573 which uses the fact that $\mathfrak{Z}$ is a meet-semilattice with least element). If $a \notin S$ then the chain S can be extended adding there element a because $\bot^{\mathfrak{A}} \sqsubset a \sqsubseteq \mathcal{X}$ for any $\mathcal{X} \in S'$ what contradicts to maximality of the chain. So $a \in S$ and consequently $a \in S'$. Obviously a is the minimal element of S'. Consequently (taking into account maximality of the chain) there is no $\mathcal{Y} \in \mathfrak{A}$ such that $\bot^{\mathfrak{A}} \sqsubset \mathcal{Y} \sqsubset a$. So a is an atomic filter. Obviously $a \sqsubseteq \mathcal{F}$.

□

Definition 577. A complete lattice is *co-compact* iff $\bigsqcap S = \bot$ for a set S of elements of this lattice implies that there is its finite subset $T \subseteq S$ such that $\bigsqcap T = \bot$.

Theorem 578. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a bounded meet-semilattice.
3°. $\mathfrak{A}$ is co-compact.

Proof.

1°⇒2°. Obvious.

2°⇒3°. Poset $\mathfrak{A}$ is complete by corollary 518.

If $\bot \in \operatorname{up} \bigsqcap^{\mathfrak{A}} S$ then there are $K_i \in \operatorname{up} \bigcup S$ such that $\bot \in \operatorname{up}(K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_n)$ that is $K_0 \sqcap^{\mathfrak{Z}} \dots \sqcap^{\mathfrak{Z}} K_n = \bot$ from which easily follows $\mathcal{F}_0 \sqcap^{\mathfrak{A}} \dots \sqcap^{\mathfrak{A}} \mathcal{F}_n = \bot$ for some $\mathcal{F}_i \in S$.

□

5.18. Separability of filters

Proposition 579. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.

3°. $\mathfrak{A}$ is strongly separable.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. By properties of stars of filters.

□

REMARK 580. [14] seems to show that the above theorem cannot be generalized for a wider class of lattices.

THEOREM 581. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $\mathfrak{A}$ is an atomistic poset.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Because (used theorem 232) $\mathfrak{A}$ is atomic (theorem 576) and separable.

□

COROLLARY 582. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $\mathfrak{A}$ is atomically separable.

PROOF. By theorem 230. □

5.19. Some Criteria

THEOREM 583. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a down-aligned, with join-closed, binarily meet-closed and separable core which is a complete boolean lattice.
4°. The following conditions are equivalent for any $\mathcal{F} \in \mathfrak{A}$:
   (a) $\mathcal{F} \in \mathfrak{Z}$;
   (b) $\forall S \in \mathscr{P}\mathfrak{A} : \left(\mathcal{F} \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \bot \Rightarrow \exists \mathcal{K} \in S : \mathcal{F} \sqcap^{\mathfrak{A}} \mathcal{K} \neq \bot\right)$;
   (c) $\forall S \in \mathscr{P}\mathfrak{Z} : \left(\mathcal{F} \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \bot \Rightarrow \exists K \in S : \mathcal{F} \sqcap^{\mathfrak{A}} K \neq \bot\right)$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. The filtrator $(\mathfrak{A}, \mathfrak{Z})$ is with with join-closed core by theorem 534, binarily meet-closed core by corollary 536, with separable core by theorem 537.
3°⇒4°.

4°a⇒4°b. Let $\mathcal{F} \in \mathfrak{Z}$. Then (taking into account the lemma 551)

$$\mathcal{F} \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \bot \Leftrightarrow \overline{\mathcal{F}} \not\supseteq \bigsqcup^{\mathfrak{A}} S \Rightarrow \exists \mathcal{K} \in S : \overline{\mathcal{F}} \not\supseteq \mathcal{K} \Leftrightarrow \exists \mathcal{K} \in S : \mathcal{F} \sqcap^{\mathfrak{A}} \mathcal{K} \neq \bot.$$

4°b⇒4°c. Obvious.

4°c⇒4°a.

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\mathcal{F} \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \bot \Rightarrow \exists K \in S : \mathcal{F} \sqcap^{\mathfrak{A}} K \neq \bot \right) \Leftrightarrow$$

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\mathcal{F} \not\asymp^{\mathfrak{A}} \bigsqcup^{\mathfrak{Z}} S \Rightarrow \exists K \in S : \mathcal{F} \not\asymp^{\mathfrak{A}} K \right) \Leftrightarrow \text{ (lemma 551)}$$

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\overline{\bigsqcup^{\mathfrak{Z}} S} \not\sqsupseteq \mathcal{F} \Rightarrow \exists K \in S : \overline{K} \not\sqsupseteq \mathcal{F} \right) \Leftrightarrow$$

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\forall K \in S : \overline{K} \sqsupseteq \mathcal{F} \Rightarrow \overline{\bigsqcup^{\mathfrak{Z}} S} \sqsupseteq \mathcal{F} \right) \Leftrightarrow$$

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\forall K \in S : \overline{K} \sqsupseteq \mathcal{F} \Rightarrow \bigsqcap^{\mathfrak{Z}} \langle \neg \rangle^* S \sqsupseteq \mathcal{F} \right) \Leftrightarrow$$

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\forall K \in S : K \sqsupseteq \mathcal{F} \Rightarrow \bigsqcap^{\mathfrak{Z}} S \sqsupseteq \mathcal{F} \right) \Rightarrow$$

$$\bigsqcap^{\mathfrak{Z}} \operatorname{up} \mathcal{F} \sqsupseteq \mathcal{F} \Leftrightarrow$$

$$\bigsqcap^{\mathfrak{Z}} \operatorname{up} \mathcal{F} \in \operatorname{up} \mathcal{F} \Rightarrow$$

$$\mathcal{F} \in \mathfrak{Z}.$$

□

Remark 584. The above theorem strengthens theorem 53 in [32]. Both the formulation of the theorem and the proof are considerably simplified.

Definition 585. Let S be a subset of a meet-semilattice. The *filter base generated by* S is the set

$$[S]_{\sqcap} = \left\{ \frac{a_0 \sqcap \cdots \sqcap a_n}{a_i \in S, n = 0, 1, \ldots} \right\}.$$

Lemma 586. The set of all finite subsets of an infinite set A has the same cardinality as A.

Proof. Let denote the number of n-element subsets of A as s_n. Obviously $s_n \leq \operatorname{card} A^n = \operatorname{card} A$. Then the number S of all finite subsets of A is equal to

$$s_0 + s_1 + \cdots \leq \operatorname{card} A + \operatorname{card} A + \cdots = \operatorname{card} A.$$

That $S \geq \operatorname{card} A$ is obvious. So $S = \operatorname{card} A$. □

Lemma 587. A filter base generated by an infinite set has the same cardinality as that set.

Proof. From the previous lemma. □

Definition 588. Let $\mathfrak{A}$ be a complete lattice. A set $S \in \mathscr{P}\mathfrak{A}$ is *filter-closed* when for every filter base $T \in \mathscr{P}S$ we have $\bigsqcap T \in S$.

Theorem 589. A subset S of a complete lattice is filter-closed iff for every nonempty chain $T \in \mathscr{P}S$ we have $\bigsqcap T \in S$.

Proof. (proof sketch by Joel David Hamkins)

⇒. Because every nonempty chain is a filter base.

$\Leftarrow$. We will assume that cardinality of a set is an ordinal defined by von Neumann cardinal assignment (what is a standard practice in ZFC). Recall that $\alpha < \beta \Leftrightarrow \alpha \in \beta$ for ordinals α, β.

We will take it as given that for every nonempty chain $T \in \mathscr{P}S$ we have $\bigsqcap T \in S$.

We will prove the following statement: If $\operatorname{card} S = n$ then S is filter closed, for any cardinal n.

Instead we will prove it not only for cardinals but for wider class of ordinals: If $\operatorname{card} S = n$ then S is filter-closed, for any ordinal n.

We will prove it using transfinite induction by n.

For finite n we have $\bigsqcap T \in S$ because $T \subseteq S$ has minimal element.

Let $\operatorname{card} T = n$ be an infinite ordinal.

Let the assumption hold for every $m \in \operatorname{card} T$.

We can assign $T = \left\{ \frac{a_\alpha}{\alpha \in \operatorname{card} T} \right\}$ for some a_α because $\operatorname{card} \operatorname{card} T = \operatorname{card} T$.

Consider $\beta \in \operatorname{card} T$.

Let $P_\beta = \left\{ \frac{a_\alpha}{\alpha \in \beta} \right\}$. Let $b_\beta = \bigsqcap P_\beta$. Obviously $b_\beta = \bigsqcap [P_\beta]_\sqcap$. We have

$$\operatorname{card} [P_\beta]_\sqcap = \operatorname{card} P_\beta = \operatorname{card} \beta < \operatorname{card} T$$

(used the lemma and von Neumann cardinal assignment). By the assumption of induction $b_\beta \in S$.

$\forall \beta \in \operatorname{card} T : P_\beta \subseteq T$ and thus $b_\beta \sqsupseteq \bigsqcap T$.

It is easy to see that the set $\left\{ \frac{P_\beta}{\beta \in \operatorname{card} T} \right\}$ is a chain. Consequently $\left\{ \frac{b_\beta}{\beta \in \operatorname{card} T} \right\}$ is a chain.

By the theorem conditions $b = \bigsqcap_{\beta \in \operatorname{card} T} b_\beta \in S$ (taken into account that $b_\beta \in S$ by the assumption of induction).

Obviously $b \sqsupseteq \bigsqcap T$.

$b \sqsubseteq b_\beta$ and so $\forall \beta \in \operatorname{card} T, \alpha \in \beta : b \sqsubseteq a_\alpha$. Let $\alpha \in \operatorname{card} T$. Then (because $\operatorname{card} T$ is a limit ordinal, see [47]) there exists $\beta \in \operatorname{card} T$ such that $\alpha \in \beta \in \operatorname{card} T$. So $b \sqsubseteq a_\alpha$ for every $\alpha \in \operatorname{card} T$. Thus $b \sqsubseteq \bigsqcap T$.

Finally $\bigsqcap T = b \in S$.

□

5.20. Co-Separability of Core

THEOREM 590. The following is an implications tuple.

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet infinite distributive complete lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is an up-aligned filtered filtrator whose core is a meet infinite distributive complete lattice.
4°. This filtrator is with co-separable core.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. It is obviously up-aligned, and filtered by theorem 534.

3°⇒4°. Our filtrator is with join-closed core (theorem 534).

Let $a, b \in \mathfrak{A}$. $\operatorname{Cor} a$ and $\operatorname{Cor} b$ exist since $\mathfrak{Z}$ is a complete lattice.

$\operatorname{Cor} a \in \operatorname{down} a$ and $\operatorname{Cor} b \in \operatorname{down} b$ by the theorem 542 since our filtrator is filtered. So we have

$$\begin{aligned}
\exists x \in \operatorname{down} a, y \in \operatorname{down} b : x \sqcup^{\mathfrak{A}} y = \top \Leftarrow \\
\operatorname{Cor} a \sqcup^{\mathfrak{A}} \operatorname{Cor} b = \top \Leftrightarrow \text{(by finite join-closedness of the core)} \\
\operatorname{Cor} a \sqcup^{\mathfrak{Z}} \operatorname{Cor} b = \top \Leftrightarrow \\
\overset{\mathfrak{Z}}{\bigsqcap} \operatorname{up} a \sqcup^{\mathfrak{Z}} \overset{\mathfrak{Z}}{\bigsqcap} \operatorname{up} b = \top \Leftrightarrow \text{(by infinite distributivity)} \\
\overset{\mathfrak{Z}}{\bigsqcap} \left\{ \frac{x \sqcup^{\mathfrak{Z}} y}{x \in \operatorname{up} a, y \in \operatorname{up} b} \right\} = \top \Leftarrow \\
\forall x \in \operatorname{up} a, y \in \operatorname{up} b : x \sqcup^{\mathfrak{Z}} y = \top \Leftrightarrow \text{(by binary join-closedness of the core)} \\
\forall x \in \operatorname{up} a, y \in \operatorname{up} b : x \sqcup^{\mathfrak{A}} y = \top \Leftarrow \\
a \sqcup^{\mathfrak{A}} b = \top.
\end{aligned}$$

□

5.21. Complements and Core Parts

LEMMA 591. If $(\mathfrak{A}, \mathfrak{Z})$ is a filtered, up-aligned filtrator with co-separable core which is a complete lattice, then for any $a, c \in \mathfrak{A}$

$$c \equiv^{\mathfrak{A}} a \Leftrightarrow c \equiv^{\mathfrak{A}} \operatorname{Cor} a.$$

PROOF.

$\Rightarrow$. If $c \equiv^{\mathfrak{A}} a$ then by co-separability of the core exists $K \in \operatorname{down} a$ such that $c \equiv^{\mathfrak{A}} K$. To finish the proof we will show that $K \sqsubseteq \operatorname{Cor} a$. To show this is enough to show that $\forall X \in \operatorname{up} a : K \sqsubseteq X$ what is obvious.

$\Leftarrow$. $\operatorname{Cor} a \sqsubseteq a$ (by theorem 542 using that our filtrator is filtered).

□

THEOREM 592. If $(\mathfrak{A}, \mathfrak{Z})$ is a filtered up-aligned complete lattice filtrator with co-separable core which is a complete boolean lattice, then $a^+ = \overline{\operatorname{Cor} a}$ for every $a \in \mathfrak{A}$.

PROOF. Our filtrator is with join-closed core (theorem 534).

$$\begin{aligned}
a^+ = \\
\overset{\mathfrak{A}}{\bigsqcap} \left\{ \frac{c \in \mathfrak{A}}{c \sqcup^{\mathfrak{A}} a = \top^{\mathfrak{A}}} \right\} = \\
\overset{\mathfrak{A}}{\bigsqcap} \left\{ \frac{c \in \mathfrak{A}}{c \sqcup^{\mathfrak{A}} \operatorname{Cor} a = \top^{\mathfrak{A}}} \right\} = \\
\overset{\mathfrak{A}}{\bigsqcap} \left\{ \frac{c \in \mathfrak{A}}{c \sqsupseteq \overline{\operatorname{Cor} a}} \right\} = \\
\overline{\operatorname{Cor} a}
\end{aligned}$$

(used the lemma above and lemma 551). □

COROLLARY 593. If $(\mathfrak{A}, \mathfrak{Z})$ is a filtered up-aligned complete lattice filtrator with co-separable core which is a complete boolean lattice, then $a^+ \in \mathfrak{Z}$ for every $a \in \mathfrak{A}$.

THEOREM 594.

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.

3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered complete lattice filtrator with down-aligned, binarily meet-closed, separable core which is a complete boolean lattice.

4°. $a^* = \overline{\operatorname{Cor} a} = \overline{\operatorname{Cor}' a}$ for every $a \in \mathfrak{A}$.

PROOF.

1°⇒2° Obvious.

2°⇒3° It is filtered by theorem 534. It is complete lattice filtrator by 518. It is with binarily meet-closed core (proposition 536), with separable core (theorem 537).

3°⇒4° Our filtrator is with join-closed core (theorem 534).

$$a^* = \bigsqcup^{\mathfrak{A}} \left\{ \frac{c \in \mathfrak{A}}{c \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\}.$$

But

$$c \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}} \Rightarrow \exists C \in \operatorname{up} c : C \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}.$$

So

$$\begin{aligned} a^* &= \\ \bigsqcup^{\mathfrak{A}} \left\{ \frac{C \in \mathfrak{Z}}{C \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\} &= \\ \bigsqcup^{\mathfrak{A}} \left\{ \frac{C \in \mathfrak{Z}}{a \sqsubseteq \overline{C}} \right\} &= \\ \bigsqcup^{\mathfrak{A}} \left\{ \frac{\overline{C}}{C \in \mathfrak{Z}, a \sqsubseteq C} \right\} &= \\ \bigsqcup^{\mathfrak{A}} \left\{ \frac{\overline{C}}{C \in \operatorname{up} a} \right\} &= \\ \overline{\bigsqcup^{\mathfrak{Z}} \left\{ \frac{\overline{C}}{C \in \operatorname{up} a} \right\}} &= \\ \overline{\bigsqcap^{\mathfrak{Z}} \left\{ \frac{C}{C \in \operatorname{up} a} \right\}} &= \\ \overline{\bigsqcap^{\mathfrak{Z}} \operatorname{up} a} &= \\ \overline{\operatorname{Cor} a} \end{aligned}$$

(used lemma 551).

$\operatorname{Cor} a = \operatorname{Cor}' a$ by theorem 545. □

THEOREM 595. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.

3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered down-aligned and up-aligned complete lattice filtrator with binarily meet-closed, separable and co-separable core which is a complete boolean lattice.

4°. $a^* = a^+ = \overline{\operatorname{Cor} a} = \overline{\operatorname{Cor}' a} \in \mathfrak{Z}$ for every $a \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. The filtrator $(\mathfrak{A}, \mathfrak{Z})$ is filtered by the theorem 534. $\mathfrak{A}$ is a complete lattice by corollary 518. $(\mathfrak{A}, \mathfrak{Z})$ is with co-separable core by theorem 590. $(\mathfrak{A}, \mathfrak{Z})$ is binarily meet-closed by proposition 536, with separable core by theorem 537.

3°⇒4°. Comparing two last theorems.

□

THEOREM 596. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete lattice.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a complete lattice filtrator with join-closed separable core which is a complete lattice.
3°. $a^* \in \mathfrak{Z}$ for every $a \in \mathfrak{A}$.

PROOF.

1°⇒2°. $\mathfrak{A}$ is a complete lattice by corollary 518. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with join-closed core by theorem 534. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with separable core by theorem 537.

2°⇒3°. $\left\{ \frac{c \in \mathfrak{A}}{c \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\} \supseteq \left\{ \frac{A \in \mathfrak{Z}}{A \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\}$; consequently $a^* \sqsupseteq \bigsqcup^{\mathfrak{A}} \left\{ \frac{A \in \mathfrak{Z}}{A \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\}$.

But if $c \in \left\{ \frac{c \in \mathfrak{A}}{c \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\}$ then there exists $A \in \mathfrak{Z}$ such that $A \sqsupseteq c$ and $A \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}$ that is $A \in \left\{ \frac{A \in \mathfrak{Z}}{A \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\}$. Consequently $a^* \sqsubseteq \bigsqcup^{\mathfrak{A}} \left\{ \frac{A \in \mathfrak{Z}}{A \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\}$.

We have $a^* = \bigsqcup^{\mathfrak{A}} \left\{ \frac{A \in \mathfrak{Z}}{A \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\} = \bigsqcup^{\mathfrak{Z}} \left\{ \frac{A \in \mathfrak{Z}}{A \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\} \in \mathfrak{Z}$.

□

THEOREM 597. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is an up-aligned filtered complete lattice filtrator with co-separable core which is a complete boolean lattice.
4°. a^+ is dual pseudocomplement of a, that is

$$a^+ = \min \left\{ \frac{c \in \mathfrak{A}}{c \sqcup^{\mathfrak{A}} a = \top^{\mathfrak{A}}} \right\}$$

for every $a \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. $(\mathfrak{A}, \mathfrak{Z})$ is filtered by the theorem 534. It is with co-separable core by theorem 590. $\mathfrak{A}$ is a complete lattice by corollary 518.

3°⇒4°. Our filtrator is with join-closed core (theorem 534). It's enough to prove that $a^+ \sqcup^{\mathfrak{A}} a = \top^{\mathfrak{A}}$. But $a^+ \sqcup^{\mathfrak{A}} a = \overline{\operatorname{Cor} a} \sqcup^{\mathfrak{A}} a \sqsupseteq \overline{\operatorname{Cor} a} \sqcup^{\mathfrak{A}} \operatorname{Cor} a = \overline{\operatorname{Cor} a} \sqcup^{\mathfrak{Z}} \operatorname{Cor} a = \top^{\mathfrak{A}}$ (used the theorem 542 and the fact that our filtrator is filtered).

□

DEFINITION 598. The *edge part* of an element $a \in \mathfrak{A}$ is $\operatorname{Edg} a = a \setminus \operatorname{Cor} a$, the *dual edge part* is $\operatorname{Edg}' a = a \setminus \operatorname{Cor}' a$.

Knowing core part and edge part or dual core part and dual edge part of an element of a filtrator, the filter can be restored by the formulas:

$$a = \operatorname{Cor} a \sqcup^{\mathfrak{A}} \operatorname{Edg} a \quad \text{and} \quad a = \operatorname{Cor}' a \sqcup^{\mathfrak{A}} \operatorname{Edg}' a.$$

5.22. Core Part and Atomic Elements

PROPOSITION 599. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over an atomistic lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with join-closed core and $\mathfrak{Z}$ be an atomistic lattice.

4°. $\operatorname{Cor}' a = \bigsqcup^{\mathfrak{Z}} \left\{ \frac{x}{x \text{ is an atom of } \mathfrak{Z}, x \sqsubseteq a} \right\}$ for every $a \in \mathfrak{A}$ such that $\operatorname{Cor}' a$ exists.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. $(\mathfrak{A}, \mathfrak{Z})$ is with join-closed core by corollary 534.

3°⇒4°.

$$\begin{aligned}
&\operatorname{Cor}' a = \\
&\bigsqcup^{\mathfrak{Z}} \left\{ \frac{A \in \mathfrak{Z}}{A \sqsubseteq a} \right\} = \\
&\bigsqcup^{\mathfrak{Z}} \left\{ \frac{\bigsqcup^{\mathfrak{Z}} \operatorname{atoms}^{\mathfrak{Z}} A}{A \in \mathfrak{Z}, A \sqsubseteq a} \right\} = \\
&\bigsqcup^{\mathfrak{Z}} \bigcup \left\{ \frac{\operatorname{atoms}^{\mathfrak{Z}} A}{A \in \mathfrak{Z}, A \sqsubseteq a} \right\} = \\
&\bigsqcup^{\mathfrak{Z}} \left\{ \frac{x}{x \text{ is an atom of } \mathfrak{Z}, x \sqsubseteq a} \right\}.
\end{aligned}$$

□

COROLLARY 600. $\operatorname{Cor} a = \uparrow \left\{ \frac{p \in \mathfrak{U}}{\uparrow\{p\} \sqsubseteq a} \right\}$ and $\bigcap a = \left\{ \frac{p \in \mathfrak{U}}{\uparrow\{p\} \sqsubseteq a} \right\}$ for every filter a on a set $\mathfrak{U}$.

PROOF. By proposition 546. □

5.23. Distributivity of Core Part over Lattice Operations

THEOREM 601. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a join-closed filtrator and $\mathfrak{A}$ is a meet-semilattice and $\mathfrak{Z}$ is a meet-semilattice.
4°. $\operatorname{Cor}'(a \sqcap^{\mathfrak{A}} b) = \operatorname{Cor}' a \sqcap^{\mathfrak{Z}} \operatorname{Cor}' b$ for every $a, b \in \mathfrak{A}$. whenever $\operatorname{Cor}'(a \sqcap^{\mathfrak{A}} b)$, $\operatorname{Cor}' a$, and $\operatorname{Cor}' b$ exist

PROOF.

1°⇒2°. Obvious.

2°⇒3°. $(\mathfrak{A}, \mathfrak{Z})$ is with join-closed core by corollary 534. $\mathfrak{A}$ is a meet-semilattice by corollary 518.

3°⇒4°. We have $\operatorname{Cor}' p \sqsubseteq p$ for every $p \in \mathfrak{A}$ whenever $\operatorname{Cor}' p$ exists, because our filtrator is with join-closed core (theorem 543).

Obviously $\operatorname{Cor}'(a \sqcap^{\mathfrak{A}} b) \sqsubseteq \operatorname{Cor}' a$ and $\operatorname{Cor}'(a \sqcap^{\mathfrak{A}} b) \sqsubseteq \operatorname{Cor}' b$.

If $x \sqsubseteq \operatorname{Cor}' a$ and $x \sqsubseteq \operatorname{Cor}' b$ for some $x \in \mathfrak{Z}$ then $x \sqsubseteq a$ and $x \sqsubseteq b$, thus $x \sqsubseteq a \sqcap^{\mathfrak{A}} b$ and $x \sqsubseteq \operatorname{Cor}'(a \sqcap^{\mathfrak{A}} b)$.

□

THEOREM 602. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a join-closed filtrator.
4°. $\operatorname{Cor}' \bigsqcap^{\mathfrak{A}} S = \bigsqcap^{\mathfrak{Z}} \langle \operatorname{Cor}' \rangle^* S$ for every $S \in \mathscr{P}\mathfrak{A}$ whenever both sides of the equality are defined. Also $\operatorname{Cor}' \bigsqcap^{\mathfrak{A}} T = \bigsqcap^{\mathfrak{Z}} T$ for every $T \in \mathscr{P}\mathfrak{Z}$ whenever both sides of the equality are defined.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. It is with join-closed core by theorem 534. $\mathfrak{A}$ is a complete lattice by corollary 518.

3°⇒4°. We have $\operatorname{Cor}' p \sqsubseteq p$ for every $p \in \mathfrak{A}$ because our filtrator is with join-closed core (theorem 543).

Obviously $\operatorname{Cor}' \bigsqcap^{\mathfrak{A}} S \sqsubseteq \operatorname{Cor}' a$ for every $a \in S$.

If $x \sqsubseteq \operatorname{Cor}' a$ for every $a \in S$ for some $x \in \mathfrak{Z}$ then $x \sqsubseteq a$, thus $x \sqsubseteq \bigsqcap^{\mathfrak{A}} S$ and $x \sqsubseteq \operatorname{Cor}' \bigsqcap^{\mathfrak{A}} S$.

So $\operatorname{Cor}' \bigsqcap^{\mathfrak{A}} S = \bigsqcap^{\mathfrak{Z}} \langle \operatorname{Cor}' \rangle^* S$. $\operatorname{Cor}' \bigsqcap^{\mathfrak{A}} T = \bigsqcap^{\mathfrak{Z}} T$ trivially follows from this.

□

THEOREM 603. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete atomistic distributive lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered down-aligned filtrator with binarily meet-closed core $\mathfrak{Z}$ which is a complete atomistic lattice and $\mathfrak{A}$ is a complete starrish lattice.
4°. $\operatorname{Cor}'(a \sqcup^{\mathfrak{A}} b) = \operatorname{Cor}' a \sqcup^{\mathfrak{Z}} \operatorname{Cor}' b$ for every $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. $(\mathfrak{A}, \mathfrak{Z})$ is filtered by theorem 534. It is with binarily meet-close core by corollary 536. $\mathfrak{A}$ is starrish by corollary 531. $\mathfrak{A}$ is complete by corollary 518.

3°⇒4°. From theorem conditions it follows that $\operatorname{Cor}'(a \sqcup^{\mathfrak{A}} b)$ exists.

$\operatorname{Cor}'(a \sqcup^{\mathfrak{A}} b) = \bigsqcup^{\mathfrak{Z}} \left\{ \frac{x}{x \text{ is an atom of } \mathfrak{Z}, x \sqsubseteq a \sqcup^{\mathfrak{A}} b} \right\}$ (used proposition 599).

By theorem 558 we have

$$\begin{aligned}
\operatorname{Cor}'(a \sqcup^{\mathfrak{A}} b) &= \\
\bigsqcup^{\mathfrak{Z}}((\operatorname{atoms}^{\mathfrak{A}}(a \sqcup^{\mathfrak{A}} b)) \cap \mathfrak{Z}) &= \\
\bigsqcup^{\mathfrak{Z}}((\operatorname{atoms}^{\mathfrak{A}} a \cup \operatorname{atoms}^{\mathfrak{A}} b) \cap \mathfrak{Z}) &= \\
\bigsqcup^{\mathfrak{Z}}((\operatorname{atoms}^{\mathfrak{A}} a \cap \mathfrak{Z}) \cup (\operatorname{atoms}^{\mathfrak{A}} b \cap \mathfrak{Z})) &= \\
\bigsqcup^{\mathfrak{Z}}(\operatorname{atoms}^{\mathfrak{A}} a \cap \mathfrak{Z}) \sqcup^{\mathfrak{Z}} \bigsqcup^{\mathfrak{Z}}(\operatorname{atoms}^{\mathfrak{A}} b \cap \mathfrak{Z})
\end{aligned}$$

(used the theorem 496). Again using theorem 558, we get

$$\begin{aligned}
&\operatorname{Cor}'(a \sqcup^{\mathfrak{A}} b) = \\
&\bigsqcup^{\mathfrak{Z}} \left\{ \frac{x}{x \text{ is an atom of } \mathfrak{Z}, x \sqsubseteq a} \right\} \sqcup^{\mathfrak{Z}} \bigsqcup^{\mathfrak{Z}} \left\{ \frac{x}{x \text{ is an atom of } \mathfrak{Z}, x \sqsubseteq b} \right\} = \\
&\operatorname{Cor}' a \sqcup^{\mathfrak{Z}} \operatorname{Cor}' b
\end{aligned}$$

(again used proposition 599).

□

See also theorem 167 above.

5.24. Separability criteria

THEOREM 604. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with correct intersection, with binarily meet-closed and separable core.
4°. $B \asymp^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \sqsupseteq \mathcal{A}$ for every $B \in \mathfrak{Z}$, $\mathcal{A} \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Using proposition 549, corollary 536, theorem 537.
3°⇒4°. By the lemma 551.

□

THEOREM 605. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator over a boolean lattice with correct joining and co-separable core.
4°. $B \equiv^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \sqsubseteq \mathcal{A}$ for every $B \in \mathfrak{Z}$, $\mathcal{A} \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Using obvious 550, theorem 590.
3°⇒4°. By the lemma 551.

□

5.25. Filtrators over Boolean Lattices

PROPOSITION 606. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a down-aligned and up-aligned binarily meet-closed and binarily join-closed distributive lattice filtrator and $\mathfrak{Z}$ is a boolean lattice.
4°. $a \setminus^{\mathfrak{A}} B = a \sqcap^{\mathfrak{A}} \overline{B}$ for every $a \in \mathfrak{A}$, $B \in \mathfrak{Z}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. $\mathfrak{A}$ is a distributive lattice by corollary 531. Our filtrator is binarily meet-closed by the corollary 536 and with join-closed core by the theorem 534. It is also up and down aligned.
3°⇒4°.

$$(a \sqcap^{\mathfrak{A}} \overline{B}) \sqcup^{\mathfrak{A}} B = (a \sqcup^{\mathfrak{A}} B) \sqcap^{\mathfrak{A}} (\overline{B} \sqcup^{\mathfrak{A}} B) =$$
$$(a \sqcup^{\mathfrak{A}} B) \sqcap^{\mathfrak{A}} (\overline{B} \sqcup^{\mathfrak{Z}} B) = (a \sqcup^{\mathfrak{A}} B) \sqcap^{\mathfrak{A}} \top = a \sqcup^{\mathfrak{A}} B.$$
$$(a \sqcap^{\mathfrak{A}} \overline{B}) \sqcap^{\mathfrak{A}} B = a \sqcap^{\mathfrak{A}} (\overline{B} \sqcap^{\mathfrak{A}} B) = a \sqcap^{\mathfrak{A}} (\overline{B} \sqcap^{\mathfrak{Z}} B) = a \sqcap^{\mathfrak{A}} \bot = \bot.$$

So $a \sqcap^{\mathfrak{A}} \overline{B}$ is the difference of a and B.

□

PROPOSITION 607. For a primary filtrator over a complete boolean lattice both edge part and dual edge part are always defined.

PROOF. Core part and dual core part are defined because the core is a complete lattice. Using the theorem 606. □

THEOREM 608. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a complete co-brouwerian atomistic down-aligned lattice filtrator with binarily meet-closed and separable boolean core.
3°. The three expressions of pseudodifference of a and b in theorem 247 are also equal to $\bigsqcup\left\{\frac{a\sqcap\overline{B}}{B\in\operatorname{up} b}\right\}$.

PROOF.

1°⇒2°. The filtrator of filters on a boolean lattice is:

- complete by corollary 518;
- atomistic by theorem 581;
- co-brouwerian by corollary 531;
- with separable core by theorem 537;
- with binarily meet-closed core by corollary 536.

2°⇒3°. $\bigsqcup\left\{\frac{z\in\mathscr{F}}{z\sqsubseteq a\wedge z\sqcap b=\bot}\right\}\sqsubseteq\bigsqcup\left\{\frac{a\sqcap\overline{B}}{B\in\operatorname{up} b}\right\}$ because

$$z\in\left\{\frac{z\in\mathscr{F}}{z\sqsubseteq a\wedge z\sqcap b=\bot}\right\}\Leftrightarrow z\sqsubseteq a\wedge z\sqcap b=\bot\Leftrightarrow(\text{separability})$$

$$z\sqsubseteq a\wedge\exists B\in\operatorname{up} b: z\sqcap B=\bot\Leftrightarrow(\text{theorem 604})\Leftrightarrow z\sqsubseteq a\wedge\exists B\in\operatorname{up} b: z\sqsubseteq\overline{B}\Leftrightarrow$$
$$\exists B\in\operatorname{up} b:\left(z\sqsubseteq a\wedge z\sqsubseteq\overline{B}\right)\Leftrightarrow\exists B\in\operatorname{up} b: z\sqsubseteq a\sqcap\overline{B}\Rightarrow$$
$$z\sqsubseteq\bigsqcup\left\{\frac{a\sqcap\overline{B}}{B\in\operatorname{up} b}\right\}.$$

But $a\sqcap\overline{B}\in\left\{\frac{z\in\mathscr{F}}{z\sqsubseteq a\wedge z\sqcap b=\bot}\right\}$ because

$$\left(a\sqcap\overline{B}\right)\sqcap b=a\sqcap\left(\overline{B}\sqcap b\right)\sqsubseteq a\sqcap\left(\overline{B}\sqcap^{\mathfrak{A}}B\right)=a\sqcap\left(\overline{B}\sqcap^{\mathfrak{Z}}B\right)=a\sqcap\bot=\bot$$

and thus

$$a\sqcap\overline{B}\sqsubseteq\bigsqcup\left\{\frac{z\in\mathscr{F}}{z\sqsubseteq a\wedge z\sqcap b=\bot}\right\}$$

so $\bigsqcup\left\{\frac{z\in\mathscr{F}}{z\sqsubseteq a\wedge z\sqcap b=\bot}\right\}\sqsupseteq\bigsqcup\left\{\frac{a\sqcap\overline{B}}{B\in\operatorname{up} b}\right\}$.

□

5.26. Distributivity for an Element of Boolean Core

LEMMA 609. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is an up-aligned binarily join-closed and binarily meet-closed distributive lattice filtrator over a boolean lattice.
4°. $A\sqcap^{\mathfrak{A}}$ is a lower adjoint of $\overline{A}\sqcup^{\mathfrak{A}}$ for every $A\in\mathfrak{Z}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. It is binarily join closed by theorem 534. It is binarily meet-closed by corollary 536. It is distributive by corollary 531.

3°⇒4°. We will use the theorem 126.

That $A\sqcap^{\mathfrak{A}}$ and $\overline{A}\sqcup^{\mathfrak{A}}$ are monotone is obvious.

We need to prove (for every $x, y\in\mathfrak{A}$) that

$$x\sqsubseteq\overline{A}\sqcup^{\mathfrak{A}}(A\sqcap^{\mathfrak{A}}x)\quad\text{and}\quad A\sqcap^{\mathfrak{A}}(\overline{A}\sqcup^{\mathfrak{A}}y)\sqsubseteq y.$$

Really,

$$\overline{A} \sqcup^{\mathfrak{A}} (A \sqcap^{\mathfrak{A}} x) = (\overline{A} \sqcup^{\mathfrak{A}} A) \sqcap^{\mathfrak{A}} (\overline{A} \sqcup^{\mathfrak{A}} x) = (\overline{A} \sqcup^{\mathfrak{Z}} A) \sqcap^{\mathfrak{A}} (\overline{A} \sqcup^{\mathfrak{A}} x) = \top \sqcap^{\mathfrak{A}} (\overline{A} \sqcup^{\mathfrak{A}} x) = \overline{A} \sqcup^{\mathfrak{A}} x \sqsupseteq x$$

and

$$A \sqcap^{\mathfrak{A}} (\overline{A} \sqcup^{\mathfrak{A}} y) = (A \sqcap^{\mathfrak{A}} \overline{A}) \sqcup^{\mathfrak{A}} (A \sqcap^{\mathfrak{A}} y) = (A \sqcap^{\mathfrak{Z}} \overline{A}) \sqcup^{\mathfrak{A}} (A \sqcap^{\mathfrak{A}} y) = \bot \sqcup^{\mathfrak{A}} (A \sqcap^{\mathfrak{A}} y) = A \sqcap^{\mathfrak{A}} y \sqsubseteq y.$$

□

THEOREM 610. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is an up-aligned binarily join-closed and binarily meet-closed distributive lattice filtrator over a boolean lattice.
4°. $A \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{A}} \langle A \sqcap^{\mathfrak{A}} \rangle^* S$ for every $A \in \mathfrak{Z}$ and every set $S \in \mathscr{P}\mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. It is binarily join-closed by theorem 534. It is binarily meet-closed by corollary 536. It is distributive by corollary 531.
3°⇒4°. Direct consequence of the lemma.

□

5.27. More about the Lattice of Filters

DEFINITION 611. Atoms of $\mathfrak{F}$ are called *ultrafilters*.

DEFINITION 612. Principal ultrafilters are also called *trivial ultrafilters*.

THEOREM 613. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. The filtrator $(\mathfrak{A}, \mathfrak{Z})$ is central.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. We can conclude that $\mathfrak{A}$ is atomically separable (the corollary 582), with separable core (the theorem 537), and with join-closed core (theorem 534), binarily meet-closed by corollary 536.

We need to prove $Z(\mathfrak{A}) = \mathfrak{Z}$.

Let $\mathcal{X} \in Z(\mathfrak{A})$. Then there exists $\mathcal{Y} \in Z(\mathfrak{A})$ such that $\mathcal{X} \sqcap^{\mathfrak{A}} \mathcal{Y} = \bot^{\mathfrak{A}}$ and $\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} = \top^{\mathfrak{A}}$. Consequently there is $X \in \operatorname{up} \mathcal{X}$ such that $X \sqcap^{\mathfrak{A}} \mathcal{Y} = \bot^{\mathfrak{A}}$; we also have $X \sqcup^{\mathfrak{A}} \mathcal{Y} = \top^{\mathfrak{A}}$. Suppose $X \sqsupset \mathcal{X}$. Then there exists $a \in \operatorname{atoms}^{\mathfrak{A}} X$ such that $a \notin \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$. We can conclude also $a \notin \operatorname{atoms}^{\mathfrak{A}} \mathcal{Y}$ (otherwise $X \sqcap^{\mathfrak{A}} \mathcal{Y} \neq \bot^{\mathfrak{A}}$). Thus $a \notin \operatorname{atoms}(\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y})$ and consequently $\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} \neq \top^{\mathfrak{A}}$ what is a contradiction. We have $\mathcal{X} = X \in \mathfrak{Z}$.

Let now $X \in \mathfrak{Z}$. Let $Y = \overline{X}$. We have $X \sqcap^{\mathfrak{Z}} Y = \bot^{\mathfrak{A}}$ and $X \sqcup^{\mathfrak{Z}} Y = \top^{\mathfrak{A}}$. Thus $X \sqcap^{\mathfrak{A}} Y = \bigsqcap^{\mathfrak{A}} \{X \sqcap^{\mathfrak{Z}} Y\} = \bot^{\mathfrak{A}}$; $X \sqcap^{\mathfrak{A}} Y = X \sqcap^{\mathfrak{Z}} Y = \top^{\mathfrak{A}}$. We have shown that $X \in Z(\mathfrak{A})$.

□

5.28. More Criteria

THEOREM 614. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. For every $S \in \mathscr{P}\mathfrak{A}$ the condition $\exists \mathcal{F} \in \mathfrak{A} : S = \star\mathcal{F}$ is equivalent to conjunction of the following items:
   (a) S is a free star on $\mathfrak{A}$;
   (b) S is filter-closed.

PROOF.

1°⇒2°. Obvious.
2°⇒3°.
⇒.
3°a. That $\bot^{\mathfrak{A}} \notin \star\mathcal{F}$ is obvious. For every $a, b \in \mathfrak{A}$

$$\begin{aligned} a \sqcup^{\mathfrak{A}} b \in \star\mathcal{F} \Leftrightarrow \\ (a \sqcup^{\mathfrak{A}} b) \sqcap^{\mathfrak{A}} \mathcal{F} \neq \bot^{\mathfrak{A}} \Leftrightarrow \\ (a \sqcap^{\mathfrak{A}} \mathcal{F}) \sqcup^{\mathfrak{A}} (b \sqcap^{\mathfrak{A}} \mathcal{F}) \neq \bot^{\mathfrak{A}} \Leftrightarrow \\ a \sqcap^{\mathfrak{A}} \mathcal{F} \neq \bot^{\mathfrak{A}} \vee b \sqcap^{\mathfrak{A}} \mathcal{F} \neq \bot^{\mathfrak{A}} \Leftrightarrow \\ a \in \star\mathcal{F} \vee \in \star\mathcal{F} \end{aligned}$$

(taken into account corollary 531). So $\star\mathcal{F}$ is a free star on $\mathfrak{A}$.

3°b. We have a filter base $T \subseteq S$ and need to prove that $\bigsqcap^{\mathfrak{A}} T \sqcap \mathcal{F} \neq \bot^{\mathfrak{A}}$. Because $\left\langle \mathcal{F} \sqcap^{\mathfrak{A}} \right\rangle^* T$ is a generalized filter base, $\bot^{\mathfrak{A}} \in \left\langle \mathcal{F} \sqcap^{\mathfrak{A}} \right\rangle^* T \Leftrightarrow \bigsqcap^{\mathfrak{A}} \left\langle \mathcal{F} \sqcap^{\mathfrak{A}} \right\rangle^* T = \bot^{\mathfrak{A}} \Leftrightarrow \bigsqcap^{\mathfrak{A}} T \sqcap^{\mathfrak{A}} \mathcal{F} \neq \bot^{\mathfrak{A}}$. So it is left to prove $\bot^{\mathfrak{A}} \notin \left\langle \mathcal{F} \sqcap^{\mathfrak{A}} \right\rangle^* T$ what follows from $T \subseteq S$.

⇐. Let S be a free star on $\mathfrak{A}$. Then for every $A, B \in \mathfrak{Z}$

$$\begin{aligned} A, B \in S \cap \mathfrak{Z} \Leftrightarrow \\ A, B \in S \Leftrightarrow \\ A \sqcup^{\mathfrak{A}} B \in S \Leftrightarrow \\ A \sqcup^{\mathfrak{Z}} B \in S \Leftrightarrow \\ A \sqcup^{\mathfrak{Z}} B \in S \cap \mathfrak{Z} \end{aligned}$$

(taken into account the theorem 534). So $S \cap \mathfrak{Z}$ is a free star on $\mathfrak{Z}$. Thus there exists $\mathcal{F} \in \mathfrak{A}$ such that $\partial\mathcal{F} = S \cap \mathfrak{Z}$. We have $\operatorname{up}\mathcal{X} \subseteq S \Leftrightarrow \mathcal{X} \in S$ (because S is filter-closed) for every $\mathcal{X} \in \mathfrak{A}$; then (taking

into account properties of generalized filter bases)

$$\begin{aligned}
\mathcal{X} \in S &\Leftrightarrow \\
\operatorname{up}\mathcal{X} \subseteq S &\Leftrightarrow \\
\operatorname{up}\mathcal{X} \subseteq \partial\mathcal{F} &\Leftrightarrow \\
\forall X \in \operatorname{up}\mathcal{X} : X \sqcap^{\mathfrak{A}} \mathcal{F} \neq \bot^{\mathfrak{A}} &\Leftrightarrow \\
\bot^{\mathfrak{A}} \notin \left\langle \mathcal{F} \sqcap^{\mathfrak{A}} \right\rangle^{*} \operatorname{up}\mathcal{X} &\Leftrightarrow \\
\bigsqcap^{\mathfrak{A}} \left\langle \mathcal{F} \sqcap^{\mathfrak{A}} \right\rangle^{*} \operatorname{up}\mathcal{X} \neq \bot^{\mathfrak{A}} &\Leftrightarrow \\
\mathcal{F} \sqcap^{\mathfrak{A}} \bigsqcap^{\mathfrak{A}} \operatorname{up}\mathcal{X} \neq \bot^{\mathfrak{A}} &\Leftrightarrow \\
\mathcal{F} \sqcap^{\mathfrak{A}} \mathcal{X} \neq \bot^{\mathfrak{A}} &\Leftrightarrow \\
\mathcal{X} \in \star\mathcal{F}.
\end{aligned}$$

□

5.29. Filters and a Special Sublattice

Remind that $Z(X)$ is the center of lattice X and Da is the lattice $\left\{ \frac{x \in \mathfrak{A}}{x \sqsubseteq a} \right\}$.

THEOREM 615. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. Let $\mathcal{A} \in \mathfrak{A}$. Then for each $\mathcal{X} \in \mathfrak{A}$

$$\mathcal{X} \in Z(D\mathcal{A}) \Leftrightarrow \exists X \in \mathfrak{Z} : \mathcal{X} = X \sqcap^{\mathfrak{A}} \mathcal{A}.$$

PROOF.

1°⇒2°. Obvious.

2°⇒3°.

⇐. Let $\mathcal{X} = X \sqcap^{\mathfrak{A}} \mathcal{A}$ where $X \in \mathfrak{Z}$. Let also $\mathcal{Y} = \overline{X} \sqcap^{\mathfrak{A}} \mathcal{A}$. Then

$$\mathcal{X} \sqcap^{\mathfrak{A}} \mathcal{Y} = X \sqcap^{\mathfrak{A}} \overline{X} \sqcap^{\mathfrak{A}} \mathcal{A} = (X \sqcap^{\mathfrak{Z}} \overline{X}) \sqcap^{\mathfrak{A}} \mathcal{A} = \bot^{\mathfrak{A}} \sqcap^{\mathfrak{A}} \mathcal{A} = \bot^{\mathfrak{A}}$$

(used corollary 536) and

$$\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} = (X \sqcup^{\mathfrak{A}} \overline{X}) \sqcap^{\mathfrak{A}} \mathcal{A} = (X \sqcup^{\mathfrak{Z}} \overline{X}) \sqcap^{\mathfrak{A}} \mathcal{A} = \top^{\mathfrak{A}} \sqcap^{\mathfrak{A}} \mathcal{A} = \mathcal{A}$$

(used theorem 534 and corollary 531). So $\mathcal{X} \in Z(D\mathcal{A})$.

⇒. Let $\mathcal{X} \in Z(D\mathcal{A})$. Then there exists $\mathcal{Y} \in Z(D\mathcal{A})$ such that $\mathcal{X} \sqcap^{\mathfrak{A}} \mathcal{Y} = \bot^{\mathfrak{A}}$ and $\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} = \mathcal{A}$. Then (used theorem 537) there exists $X \in \operatorname{up}\mathcal{X}$ such that $X \sqcap^{\mathfrak{A}} \mathcal{Y} = \bot^{\mathfrak{A}}$. We have

$$\mathcal{X} = \mathcal{X} \sqcup (X \sqcap^{\mathfrak{A}} \mathcal{Y}) = X \sqcap^{\mathfrak{A}} (\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y}) = X \sqcap^{\mathfrak{A}} \mathcal{A}.$$

□

THEOREM 616. The following is an implication tuple:

1°. $(\mathfrak{A}; \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}; \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $\mathfrak{F}(Z(D\mathcal{A}))$ is order-isomorphic to $D\mathcal{A}$ by the formulas
   - $\mathcal{Y} = \bigsqcap \mathcal{X}$ for every $\mathcal{X} \in \mathfrak{F}(Z(D\mathcal{A}))$;
   - $\mathcal{X} = \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \sqsupseteq \mathcal{Y}} \right\}$ for every $\mathcal{Y} \in D\mathcal{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. We need to prove that the above formulas define a bijection, then it becomes evident that it's an order isomorphism (take into account that the order of filters is *reverse* to set inclusion).

First prove that these formulas describe correspondences between $\mathfrak{F}(Z(D\mathcal{A}))$ and $D\mathcal{A}$.

Let $\mathcal{X} \in \mathfrak{F}(Z(D\mathcal{A}))$. Consider $\mathcal{Y} = \bigsqcap \mathcal{X}$. Every element of $\mathcal{X}$ is below $\mathcal{A}$, consequently $\mathcal{Y} \in D\mathcal{A}$.

Let now $\mathcal{Y} \in D\mathcal{A}$. Then $\left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \sqsupseteq \mathcal{Y}} \right\}$ is a filter.

It remains to prove that these correspondences are mutually inverse.

Let $\mathcal{X} = \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \sqsupseteq \mathcal{Y}_0} \right\}$ and $\mathcal{Y}_1 = \bigsqcap \mathcal{X}$ for some $\mathcal{Y}_0 \in D\mathcal{A}$.

$\mathcal{Y}_1 \sqsupseteq \mathcal{Y}_0$ is obvious. By theorem 615 and the condition 2° we have $\mathcal{Y}_1 = \bigsqcap \mathcal{X} \sqsubseteq \bigsqcap^{\mathfrak{A}} \left\{ \frac{F \sqcap \mathcal{A}}{F \in \operatorname{up} \mathcal{Y}_0} \right\} = \bigsqcap^{\mathfrak{A}} \left\{ \frac{F}{F \in \operatorname{up} \mathcal{Y}_0} \right\} \sqcap \mathcal{A} = \mathcal{Y}_0 \sqcap \mathcal{A} = \mathcal{Y}_0$. So $\mathcal{Y}_1 = \mathcal{Y}_0$.

Let now $\mathcal{Y} = \bigsqcap \mathcal{X}_0$ and $\mathcal{X}_1 = \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \sqsupseteq \mathcal{Y}} \right\}$ for some $\mathcal{X}_0 \in \mathfrak{F}(Z(D\mathcal{A}))$.

$\mathcal{X}_1 = \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \sqsupseteq \bigsqcap \mathcal{X}_0} \right\} =$ (by generalized filter bases) $= \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\exists X \in \mathcal{X}_0 : \mathcal{F} \sqsupseteq X} \right\} = \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \in \mathcal{X}_0} \right\} = \mathcal{X}_0$ because $\mathcal{F} \in \mathcal{X}_0 \Leftrightarrow \exists X \in \mathcal{X}_0 : \mathcal{F} \sqsupseteq X$ if $\mathcal{F} \in Z(D\mathcal{A})$.

□

5.30. Distributivity of quasicomplements

THEOREM 617. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered down-aligned and up-aligned complete lattice filtrator with binarily meet-closed, separable and co-separable core which is a complete boolean lattice.
4°. $(a \sqcap^{\mathfrak{A}} b)^* = (a \sqcap^{\mathfrak{A}} b)^+ = a^* \sqcup^{\mathfrak{A}} b^* = a^+ \sqcup^{\mathfrak{A}} b^+$ for every $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. The filtrator $(\mathfrak{A}, \mathfrak{Z})$ is filtered by the theorem 534. $\mathfrak{A}$ is a complete lattice by corollary 518. $(\mathfrak{A}, \mathfrak{Z})$ is with co-separable core by theorem 590. $(\mathfrak{A}, \mathfrak{Z})$ is binarily meet-closed by proposition 536, with separable core by theorem 537.

3°⇒4°. Theorem 595 apply. Also theorem 601 apply because every filtered filtrator is join-closed. So

$$(a \sqcap^{\mathfrak{A}} b)^* = (a \sqcap^{\mathfrak{A}} b)^+ = \overline{\operatorname{Cor}(a \sqcap^{\mathfrak{A}} b)} = \overline{\operatorname{Cor} a \sqcap^{\mathfrak{Z}} \operatorname{Cor} b} = \overline{\operatorname{Cor} a} \sqcup^{\mathfrak{A}} \overline{\operatorname{Cor} b} = a^+ \sqcup^{\mathfrak{A}} b^+ = a^* \sqcup^{\mathfrak{A}} b^*.$$

□

THEOREM 618. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered starrish down-aligned and up-aligned complete lattice filtrator with binarily meet-closed, separable and co-separable core which is a complete atomistic boolean lattice.
3°. $(a \sqcup^{\mathfrak{A}} b)^* = (a \sqcup^{\mathfrak{A}} b)^+ = a^* \sqcap^{\mathfrak{A}} b^* = a^+ \sqcap^{\mathfrak{A}} b^+$ for every $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered (theorem 534), distributive (corollary 531) complete lattice filtrator (corollary 518), with binarily meet-closed core (corollary 536), with separable core (theorem 537), with co-separable core (theorem 590).

2°⇒3°. $(a \sqcup^{\mathfrak{A}} b)^+ = (a \sqcup^{\mathfrak{A}} b)^* = \overline{\operatorname{Cor}'(a \sqcup^{\mathfrak{A}} b)} = \overline{\operatorname{Cor}' a \sqcup^{\mathfrak{Z}} \operatorname{Cor}' b} = \overline{\operatorname{Cor}' a} \sqcap^{\mathfrak{Z}} \overline{\operatorname{Cor}' b} = a^* \sqcap^{\mathfrak{Z}} b^* = a^* \sqcap^{\mathfrak{A}} b^* = a^+ \sqcap^{\mathfrak{A}} b^+$ (used theorems 594, 603, 595).

□

THEOREM 619. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered complete lattice filtrator with down-aligned, binarily meet-closed, separable core which is a complete boolean lattice.
4°. $(a \sqcap^{\mathfrak{A}} b)^* = a^* \sqcup^{\mathfrak{A}} b^*$ for every $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. It is filtered by theorem 534. It is complete lattice filtrator by 518. It is with binarily meet-closed core (corollary 536), with separable core (theorem 537).

3°⇒4°. It is join closed because it is filtered.

$$(a \sqcap^{\mathfrak{A}} b)^* = \overline{\operatorname{Cor}'(a \sqcap^{\mathfrak{A}} b)} = \overline{\operatorname{Cor}' a \sqcap^{\mathfrak{Z}} \operatorname{Cor}' b} = \overline{\operatorname{Cor}' a} \sqcup^{\mathfrak{Z}} \overline{\operatorname{Cor}' b} = a^* \sqcup^{\mathfrak{Z}} b^* = a^* \sqcup^{\mathfrak{A}} b^*$$

(theorems 601, 594).

□

THEOREM 620. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered starrish down-aligned complete lattice filtrator with binarily meet-closed, separable core which is a complete atomistic boolean lattice.
3°. $(a \sqcup^{\mathfrak{A}} b)^* = a^* \sqcap^{\mathfrak{A}} b^*$ for every $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered (theorem 534), distributive (corollary 531) complete lattice filtrator (corollary 518), with binarily meet-closed core (corollary 536), with separable core (theorem 537).

2°⇒3°.

$$(a \sqcup^{\mathfrak{A}} b)^* = \overline{\operatorname{Cor}'(a \sqcup^{\mathfrak{A}} b)} = \overline{\operatorname{Cor}' a \sqcup^{\mathfrak{Z}} \operatorname{Cor}' b} = \overline{\operatorname{Cor}' a} \sqcap^{\mathfrak{Z}} \overline{\operatorname{Cor}' b} = a^* \sqcap^{\mathfrak{Z}} b^* = a^* \sqcap^{\mathfrak{A}} b^*$$

(used theorems 594, 603).

□

THEOREM 621. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered up-aligned complete lattice filtrator with co-separable core which is a complete boolean lattice.
4°. $(a \sqcap^{\mathfrak{A}} b)^+ = a^+ \sqcup^{\mathfrak{A}} b^+$ for every $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. It is filtered by theorem 534, is a complete lattice by corollary 518, is with co-separable core by theorem 590.

3°⇒4°.

$$(a \sqcap^{\mathfrak{A}} b)^+ = \overline{\mathrm{Cor}(a \sqcap^{\mathfrak{A}} b)} = \overline{\mathrm{Cor}'(a \sqcap^{\mathfrak{A}} b)} = \overline{\mathrm{Cor}'\, a \sqcap^{\mathfrak{Z}} \mathrm{Cor}'\, b} = \overline{\mathrm{Cor}'\, a} \sqcup^{\mathfrak{Z}} \overline{\mathrm{Cor}'\, b} = \overline{\mathrm{Cor}'\, a} \sqcup^{\mathfrak{A}} \overline{\mathrm{Cor}'\, b} = a^+ \sqcup^{\mathfrak{A}} b^+$$

using theorems 592, 545, 601 and the fact that filtered filtrator is join-closed.

□

THEOREM 622. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered down-aligned and up-aligned filtrator with binarily meet-closed core, with co-separable core $\mathfrak{Z}$ which is a complete atomistic boolean lattice and $\mathfrak{A}$ is a complete starrish lattice.
3°. $(a \sqcup^{\mathfrak{A}} b)^+ = a^+ \sqcap^{\mathfrak{A}} b^+$ for every $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°.

$$(a \sqcup^{\mathfrak{A}} b)^+ = \overline{\mathrm{Cor}(a \sqcup^{\mathfrak{A}} b)} = \overline{\mathrm{Cor}'(a \sqcup^{\mathfrak{A}} b)} = \overline{\mathrm{Cor}'\, a \sqcup^{\mathfrak{Z}} \mathrm{Cor}'\, b} = \overline{\mathrm{Cor}'\, a} \sqcap^{\mathfrak{Z}} \overline{\mathrm{Cor}'\, b} = \overline{\mathrm{Cor}\, a} \sqcap^{\mathfrak{A}} \overline{\mathrm{Cor}\, b} = a^+ \sqcap^{\mathfrak{A}} b^+$$

using theorems 592, 545, 603.

□

5.31. Complementive Filters and Factoring by a Filter

DEFINITION 623. Let $\mathfrak{A}$ be a meet-semilattice and $\mathcal{A} \in \mathfrak{A}$. The relation $\sim$ on $\mathfrak{A}$ is defined by the formula

$$\forall X, Y \in \mathfrak{A} : (X \sim Y \Leftrightarrow X \sqcap^{\mathfrak{A}} \mathcal{A} = Y \sqcap^{\mathfrak{A}} \mathcal{A}).$$

PROPOSITION 624. The relation $\sim$ is an equivalence relation.

PROOF.

Reflexivity. Obvious.
Symmetry. Obvious.
Transitivity. Obvious.

□

DEFINITION 625. When $X, Y \in \mathfrak{Z}$ and $\mathcal{A} \in \mathfrak{A}$ we define $X \sim Y \Leftrightarrow \uparrow X \sim \uparrow Y$.

THEOREM 626. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice.
3°. For every $\mathcal{A} \in \mathfrak{A}$ and $X, Y \in \mathfrak{Z}$ we have

$$X \sim Y \Leftrightarrow \exists A \in \mathrm{up}\, \mathcal{A} : X \sqcap^{\mathfrak{Z}} A = Y \sqcap^{\mathfrak{Z}} A.$$

PROOF.

1°⇒2°. Obvious.

2°⇒3°.

$$\begin{aligned}
\exists A \in \operatorname{up}\mathcal{A} : X \sqcap^{\mathfrak{Z}} A = Y \sqcap^{\mathfrak{Z}} A &\Leftrightarrow \text{ (corollary 536)}\\
\exists A \in \operatorname{up}\mathcal{A} : \uparrow X \sqcap^{\mathfrak{A}} \uparrow A = \uparrow Y \sqcap^{\mathfrak{A}} \uparrow A &\Rightarrow\\
\exists A \in \operatorname{up}\mathcal{A} : \uparrow X \sqcap^{\mathfrak{A}} \uparrow A \sqcap^{\mathfrak{A}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{A}} \uparrow A \sqcap^{\mathfrak{A}} \mathcal{A} &\Leftrightarrow\\
\exists A \in \operatorname{up}\mathcal{A} : \uparrow X \sqcap^{\mathfrak{A}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{A}} \mathcal{A} &\Leftrightarrow\\
\uparrow X \sqcap^{\mathfrak{A}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{A}} \mathcal{A} &\Leftrightarrow\\
\uparrow X \sim \uparrow Y &\Leftrightarrow\\
X \sim Y.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\uparrow X \sqcap^{\mathfrak{A}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{A}} \mathcal{A} &\Leftrightarrow\\
\left\{ \frac{X \sqcap^{\mathfrak{Z}} A_0}{A_0 \in \mathcal{A}} \right\} = \left\{ \frac{Y \sqcap^{\mathfrak{Z}} A_1}{A_1 \in \mathcal{A}} \right\} &\Rightarrow\\
\exists A_0, A_1 \in \operatorname{up}\mathcal{A} : X \sqcap^{\mathfrak{Z}} A_0 = Y \sqcap^{\mathfrak{Z}} A_1 &\Rightarrow\\
\exists A_0, A_1 \in \operatorname{up}\mathcal{A} : X \sqcap^{\mathfrak{Z}} A_0 \sqcap^{\mathfrak{Z}} A_1 = Y \sqcap^{\mathfrak{Z}} A_0 \sqcap^{\mathfrak{Z}} A_1 &\Rightarrow\\
\exists A \in \operatorname{up}\mathcal{A} : Y \sqcap^{\mathfrak{Z}} A = X \sqcap^{\mathfrak{Z}} A.
\end{aligned}$$

□

PROPOSITION 627. The relation $\sim$ is a congruence[1] for each of the following:

1°. a meet-semilattice $\mathfrak{A}$;
2°. a distributive lattice $\mathfrak{A}$.

PROOF. Let $a_0, a_1, b_0, b_1 \in \mathfrak{A}$ and $a_0 \sim a_1$ and $b_0 \sim b_1$.

1°. $a_0 \sqcap b_0 \sim a_1 \sqcap b_1$because $(a_0 \sqcap b_0) \sqcap \mathcal{A} = a_0 \sqcap (b_0 \sqcap \mathcal{A}) = a_0 \sqcap (b_1 \sqcap \mathcal{A}) = b_1 \sqcap (a_0 \sqcap \mathcal{A}) = b_1 \sqcap (a_1 \sqcap \mathcal{A}) = (a_1 \sqcap b_1) \sqcap \mathcal{A}$.

2°. Taking the above into account, we need to prove only $a_0 \sqcup b_0 \sim a_1 \sqcup b_1$. We have

$$(a_0 \sqcup b_0) \sqcap \mathcal{A} = (a_0 \sqcap \mathcal{A}) \sqcup (b_0 \sqcap \mathcal{A}) = (a_1 \sqcap \mathcal{A}) \sqcup (b_1 \sqcap \mathcal{A}) = (a_1 \sqcup b_1) \sqcap \mathcal{A}.$$

□

DEFINITION 628. We will denote $A/(\sim) = A/((\sim) \cap A \times A)$ for a set A and an equivalence relation $\sim$ on a set $B \supseteq A$. I will call $\sim$ a congruence on A when $(\sim) \cap (A \times A)$ is a congruence on A.

THEOREM 629. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. Let $\mathcal{A} \in \mathfrak{A}$. Consider the function $\gamma : Z(D\mathcal{A}) \to \mathfrak{Z}/\sim$ defined by the formula (for every $p \in Z(D\mathcal{A})$)

$$\gamma p = \left\{ \frac{X \in \mathfrak{Z}}{X \sqcap^{\mathfrak{A}} \mathcal{A} = p} \right\}.$$

Then:

(a) γ is a lattice isomorphism.
(b) $\forall Q \in q : \gamma^{-1} q = Q \sqcap^{\mathfrak{A}} \mathcal{A}$ for every $q \in \mathfrak{Z}/\sim$.

PROOF.

1°⇒2°. Obvious.

[1] See Wikipedia for a definition of congruence.

2°⇒3°. $\forall p \in Z(D\mathcal{A}) : \gamma p \neq \emptyset$ because of theorem 615. Thus it is easy to see that $\gamma p \in \mathfrak{Z}/\sim$ and that γ is an injection.

Let's prove that γ is a lattice homomorphism:

$\gamma(p_0 \sqcap^{\mathfrak{A}} p_1) = \left\{ \frac{X \in \mathfrak{Z}}{X \sqcap^{\mathfrak{A}} \mathcal{A} = p_0 \sqcap^{\mathfrak{A}} p_1} \right\}$;

$$\begin{aligned}
\gamma p_0 \sqcap^{\mathfrak{Z}/\sim} \gamma p_1 &= \\
\left\{ \frac{X_0 \in \mathfrak{Z}}{X_0 \sqcap^{\mathfrak{A}} \mathcal{A} = p_0} \right\} \sqcap^{\mathfrak{Z}/\sim} \left\{ \frac{X_1 \in \mathfrak{Z}}{X_1 \sqcap^{\mathfrak{A}} \mathcal{A} = p_1} \right\} &= \\
\left\{ \frac{X_0 \sqcap^{\mathfrak{A}} X_1}{X_0, X_1 \in \mathfrak{Z}, X_0 \sqcap^{\mathfrak{A}} \mathcal{A} = p_0 \wedge X_1 \sqcap^{\mathfrak{A}} \mathcal{A} = p_1} \right\} &\subseteq \\
\left\{ \frac{X' \in \mathfrak{Z}}{X' \sqcap^{\mathfrak{A}} \mathcal{A} = p_0 \sqcap^{\mathfrak{A}} p_1} \right\} &= \\
\gamma(p_0 \sqcap^{\mathfrak{A}} p_1).&
\end{aligned}$$

Because $\gamma p_0 \sqcap^{\mathfrak{Z}/\sim} \gamma p_1$ and $\gamma(p_0 \sqcap^{\mathfrak{A}} p_1)$ are equivalence classes, thus follows $\gamma p_0 \sqcap^{\mathfrak{Z}/\sim} \gamma p_1 = \gamma(p_0 \sqcap^{\mathfrak{A}} p_1)$.

To finish the proof it is enough to show that $\forall Q \in q : q = \gamma(Q \sqcap^{\mathfrak{A}} \mathcal{A})$ for every $q \in \mathfrak{Z}/\sim$. (From this it follows that γ is surjective because q is not empty and thus $\exists Q \in q : q = \gamma(Q \sqcap^{\mathfrak{A}} \mathcal{A})$.) Really,

$$\gamma(Q \sqcap^{\mathfrak{A}} \mathcal{A}) = \left\{ \frac{X \in \mathfrak{Z}}{X \sqcap^{\mathfrak{A}} \mathcal{A} = Q \sqcap^{\mathfrak{A}} \mathcal{A}} \right\} = [Q] = q.$$

□

This isomorphism is useful in both directions to reveal properties of both lattices $Z(D\mathcal{A})$ and $q \in \mathfrak{Z}/\sim$.

Corollary 630. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $\mathfrak{Z}/\sim$ is a boolean lattice

Proof. Because $Z(D\mathcal{A})$ is a boolean lattice (theorem 98). □

5.32. Pseudodifference of filters

Proposition 631. The following is an implications tuple:

1°. $\mathfrak{A}$ is a lattice of filters on a set.
2°. $\mathfrak{A}$ is a lattice of filters over a boolean lattice.
3°. $\mathfrak{A}$ is an atomistic co-brouwerian lattice.
4°. For every $a, b \in \mathfrak{A}$ the following expressions are always equal:
   (a) $a \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$ (quasidifference of a and b);
   (b) $a \# b = \bigsqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$ (second quasidifference of a and b);
   (c) $\bigsqcup(\operatorname{atoms} a \setminus \operatorname{atoms} b)$.

Proof.

1°⇒2°. Obvious.
2°⇒3°. By corollary 531 and theorem 581.
3°⇒4°. Theorem 247.

Conjecture 632. $a \setminus^* b = a \# b$ for arbitrary filters a, b on powersets is not provable in ZF (without axiom of choice).

□

5.33. Function spaces of posets

DEFINITION 633. Let $\mathfrak{A}_i$ be a family of posets indexed by some set $\operatorname{dom}\mathfrak{A}$. We will define order of indexed families of elements of posets by the formula

$$a \sqsubseteq b \Leftrightarrow \forall i \in \operatorname{dom}\mathfrak{A} : a_i \sqsubseteq b_i.$$

I will call this new poset $\prod\mathfrak{A}$ *the function space* of posets and the above order *product order*.

PROPOSITION 634. The function space for posets is also a poset.

PROOF.
Reflexivity. Obvious.
Antisymmetry. Obvious.
Transitivity. Obvious.

□

OBVIOUS 635. $\mathfrak{A}$ has least element iff each $\mathfrak{A}_i$ has a least element. In this case

$$\bot^{\prod\mathfrak{A}} = \prod_{i\in\operatorname{dom}\mathfrak{A}} \bot^{\mathfrak{A}_i}.$$

PROPOSITION 636. $a \not\asymp b \Leftrightarrow \exists i \in \operatorname{dom}\mathfrak{A} : a_i \not\asymp b_i$ for every $a, b \in \prod\mathfrak{A}$ if every $\mathfrak{A}_i$ has least element.

PROOF. If $\operatorname{dom}\mathfrak{A} = \emptyset$, then $a = b = \bot$, $a \asymp b$ and thus the theorem statement holds. Assume $\operatorname{dom}\mathfrak{A} \neq \emptyset$.

$$\begin{aligned} a \not\asymp b \Leftrightarrow \\ \exists c \in \prod\mathfrak{A} \setminus \{\bot^{\prod\mathfrak{A}}\} : (c \sqsubseteq a \wedge c \sqsubseteq b) \Leftrightarrow \\ \exists c \in \prod\mathfrak{A} \setminus \{\bot^{\prod\mathfrak{A}}\} \forall i \in \operatorname{dom}\mathfrak{A} : (c_i \sqsubseteq a_i \wedge c_i \sqsubseteq b_i) \Leftrightarrow \\ \text{(for the reverse implication take } c_j = \bot^{\mathfrak{A}_j} \text{ for } i \neq j) \\ \exists i \in \operatorname{dom}\mathfrak{A}, c \in \mathfrak{A}_i \setminus \{\bot^{\mathfrak{A}_i}\} : (c \sqsubseteq a_i \wedge c \sqsubseteq b_i) \Leftrightarrow \\ \exists i \in \operatorname{dom}\mathfrak{A} : a_i \not\asymp b_i. \end{aligned}$$

□

PROPOSITION 637.

1°. If $\mathfrak{A}_i$ are join-semilattices then $\mathfrak{A}$ is a join-semilattice and

$$A \sqcup B = \lambda i \in \operatorname{dom}\mathfrak{A} : Ai \sqcup Bi. \tag{2}$$

2°. If $\mathfrak{A}_i$ are meet-semilattices then $\mathfrak{A}$ is a meet-semilattice and

$$A \sqcap B = \lambda i \in \operatorname{dom}\mathfrak{A} : Ai \sqcap Bi.$$

PROOF. It is enough to prove the formula (2).
It's obvious that $\lambda i \in \operatorname{dom}\mathfrak{A} : Ai \sqcup Bi \sqsupseteq A, B$.
Let $C \sqsupseteq A, B$. Then (for every $i \in \operatorname{dom}\mathfrak{A}$) $Ci \sqsupseteq Ai$ and $Ci \sqsupseteq Bi$. Thus $Ci \sqsupseteq Ai \sqcup Bi$ that is $C \sqsupseteq \lambda i \in \operatorname{dom}\mathfrak{A} : Ai \sqcup Bi$. □

COROLLARY 638. If $\mathfrak{A}_i$ are lattices then $\prod\mathfrak{A}$ is a lattice.

OBVIOUS 639. If $\mathfrak{A}_i$ are distributive lattices then $\prod\mathfrak{A}$ is a distributive lattice.

PROPOSITION 640. If $\mathfrak{A}_i$ are boolean lattices then $\prod\mathfrak{A}$ is a boolean lattice.

PROOF. We need to prove only that every element $a \in \prod\mathfrak{A}$ has a complement. But this complement is evidently $\lambda i \in \operatorname{dom} a : \overline{a_i}$. □

PROPOSITION 641. If every $\mathfrak{A}_i$ is a poset then for every $S \in \mathscr{P} \prod \mathfrak{A}$

1°. $\bigsqcup S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup_{x \in S} x_i$ whenever every $\bigsqcup_{x \in S} x_i$ exists;
2°. $\bigsqcap S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcap_{x \in S} x_i$ whenever every $\bigsqcap_{x \in S} x_i$ exists.

PROOF. It's enough to prove the first formula.

$\left(\lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup_{x \in S} x_i\right)_i = \bigsqcup_{x \in S} x_i \sqsupseteq x_i$ for every $x \in S$ and $i \in \operatorname{dom} \mathfrak{A}$.

Let $y \sqsupseteq x$ for every $x \in S$. Then $y_i \sqsupseteq x_i$ for every $i \in \operatorname{dom} \mathfrak{A}$ and thus $y_i \sqsupseteq \bigsqcup_{x \in S} x_i = \left(\lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup_{x \in S} x_i\right)_i$ that is $y \sqsupseteq \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup_{x \in S} x_i$.

Thus $\bigsqcup S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup_{x \in S} x_i$ by the definition of join. □

COROLLARY 642. If $\mathfrak{A}_i$ are posets then for every $S \in \mathscr{P} \prod \mathfrak{A}$

1°. $\bigsqcup S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup_{x \in S} x_i$ whenever $\bigsqcup S$ exists;
2°. $\bigsqcap S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcap_{x \in S} x_i$ whenever $\bigsqcap S$ exists.

PROOF. It is enough to prove that (for every i) $\bigsqcup_{x \in S} x_i$ exists whenever $\bigsqcup S$ exists.

Fix $i \in \operatorname{dom} \mathfrak{A}$.

Take $y_i = (\bigsqcup S)_i$ and let prove that y_i is the least upper bound of $\left\{ \frac{x_i}{x \in S} \right\}$.

y_i is it's upper bound because $\bigsqcup S \sqsupseteq x$ and thus $(\bigsqcup S)_i \sqsupseteq x_i$ for every $x \in S$.

Let $x \in S$ and for some $t \in \mathfrak{A}_i$

$$T(t) = \lambda j \in \operatorname{dom} \mathfrak{A} : \begin{cases} t & \text{if } i = j \\ x_j & \text{if } i \neq j. \end{cases}$$

Let $t \sqsupseteq x_i$. Then $T(t) \sqsupseteq x$ for every $x \in S$. So $T(t) \sqsupseteq \bigsqcup S$ and consequently $t = T(t)_i \sqsupseteq y_i$.

So y_i is the least upper bound of $\left\{ \frac{x_i}{x \in S} \right\}$. □

COROLLARY 643. If $\mathfrak{A}_i$ are complete lattices then $\mathfrak{A}$ is a complete lattice.

OBVIOUS 644. If $\mathfrak{A}_i$ are complete (co-)brouwerian lattices then $\mathfrak{A}$ is a (co-)brouwerian lattice.

PROPOSITION 645. If each $\mathfrak{A}_i$ is a separable poset with least element (for some index set n) then $\prod \mathfrak{A}$ is a separable poset.

PROOF. Let $a \neq b$. Then $\exists i \in \operatorname{dom} \mathfrak{A} : a_i \neq b_i$. So $\exists x \in \mathfrak{A}_i : (x \not\asymp a_i \wedge x \asymp b_i)$ (or vice versa).

Take $y = \lambda j \in \operatorname{dom} \mathfrak{A} : \begin{cases} x & \text{if } j = i; \\ \perp^{\mathfrak{A}_j} & \text{if } j \neq i. \end{cases}$ Then $y \not\asymp a$ and $y \asymp b$. □

OBVIOUS 646. If every $\mathfrak{A}_i$ is a poset with least element, then the set of atoms of $\prod \mathfrak{A}$ is

$$\left\{ \frac{\lambda i \in \operatorname{dom} \mathfrak{A} : \left(\begin{cases} a & \text{if } i = k; \\ \perp^{\mathfrak{A}_i} & \text{if } i \neq k \end{cases} \right)}{k \in \operatorname{dom} \mathfrak{A}, a \in \operatorname{atoms}^{\mathfrak{A}_k}} \right\}.$$

PROPOSITION 647. If every $\mathfrak{A}_i$ is an atomistic poset with least element, then $\prod \mathfrak{A}$ is an atomistic poset.

PROOF. $x_i = \bigsqcup \operatorname{atoms} x_i$ for every $x_i \in \mathfrak{A}_i$. Thus

$$x = \lambda i \in \operatorname{dom} x : x_i = \lambda i \in \operatorname{dom} x : \bigsqcup \operatorname{atoms} x_i =$$

$$\bigsqcup_{i \in \operatorname{dom} x} \lambda j \in \operatorname{dom} x : \begin{cases} x_i & \text{if } j = i \\ \bot^{\mathfrak{A}_j} & \text{if } j \neq i \end{cases} =$$

$$\bigsqcup_{i \in \operatorname{dom} x} \lambda j \in \operatorname{dom} x : \begin{cases} \bigsqcup \operatorname{atoms} x_i & \text{if } j = i \\ \bot^{\mathfrak{A}_j} & \text{if } j \neq i \end{cases} =$$

$$\bigsqcup_{i \in \operatorname{dom} x} \bigsqcup_{q \in \operatorname{atoms} x_i} \lambda j \in \operatorname{dom} x : \begin{cases} q & \text{if } j = i \\ \bot^{\mathfrak{A}_j} & \text{if } j \neq i \end{cases}.$$

Thus x is a join of atoms of $\prod \mathfrak{A}$. □

COROLLARY 648. If $\mathfrak{A}_i$ are atomistic posets with least elements, then $\prod \mathfrak{A}$ is atomically separable.

PROOF. Proposition 230. □

PROPOSITION 649. Let $(\mathfrak{A}_{i \in n}, \mathfrak{Z}_{i \in n})$ be a family of filtrators. Then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a filtrator.

PROOF. We need to prove that $\prod \mathfrak{Z}$ is a sub-poset of $\prod \mathfrak{A}$. First $\prod \mathfrak{Z} \subseteq \prod \mathfrak{A}$ because $\mathfrak{Z}_i \subseteq \mathfrak{A}_i$ for each $i \in n$.

Let $A, B \in \prod \mathfrak{Z}$ and $A \sqsubseteq^{\prod \mathfrak{Z}} B$. Then $\forall i \in n : A_i \sqsubseteq^{\mathfrak{Z}_i} B_i$; consequently $\forall i \in n : A_i \sqsubseteq^{\mathfrak{A}_i} B_i$ that is $A \sqsubseteq^{\prod \mathfrak{A}} B$. □

PROPOSITION 650. Let $(\mathfrak{A}_{i \in n}, \mathfrak{Z}_{i \in n})$ be a family of filtrators.

1°. The filtrator $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is (binarily) join-closed if every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is (binarily) join-closed.
2°. The filtrator $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is (binarily) meet-closed if every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is (binarily) meet-closed.

PROOF. Let every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be binarily join-closed. Let $A, B \in \prod \mathfrak{Z}$ and $A \sqcup^{\prod \mathfrak{Z}} B$ exist. Then (by corollary 642)

$$A \sqcup^{\prod \mathfrak{Z}} B = \lambda i \in n : A_i \sqcup^{\mathfrak{Z}_i} B_i = \lambda i \in n : A_i \sqcup^{\mathfrak{A}_i} B_i = A \sqcup^{\prod \mathfrak{A}} B.$$

Let now every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be join-closed. Let $S \in \mathscr{P} \prod \mathfrak{Z}$ and $\bigsqcup^{\prod \mathfrak{Z}} S$ exist. Then (by corollary 642)

$$\bigsqcup^{\prod \mathfrak{Z}} S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup^{\mathfrak{Z}_i} \left\{ \frac{x_i}{x \in S} \right\} = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup^{\mathfrak{A}_i} \left\{ \frac{x_i}{x \in S} \right\} = \bigsqcup^{\prod \mathfrak{A}} S.$$

The rest follows from symmetry. □

PROPOSITION 651. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ where $i \in n$ (for some index set n) is a down-aligned filtrator with separable core then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is with separable core.

PROOF. Let $a \neq b$. Then $\exists i \in n : a_i \neq b_i$. So $\exists x \in \mathfrak{Z}_i : (x \not\asymp a_i \wedge x \asymp b_i)$ (or vice versa).

Take $y = \lambda j \in n : \begin{cases} x & \text{if } j = i \\ \bot^{\mathfrak{A}_j} & \text{if } j \neq i \end{cases}$. Then we have $y \not\asymp a$ and $y \asymp b$ and $y \in \mathfrak{Z}$. □

PROPOSITION 652. Let every $\mathfrak{A}_i$ be a bounded lattice. Every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a central filtrator iff $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a central filtrator.

Proof.

$$\begin{aligned} x \in Z\Big(\prod \mathfrak{A}\Big) &\Leftrightarrow \\ \exists y \in \prod \mathfrak{A} : (x \sqcap y = \bot^{\prod \mathfrak{A}} \wedge x \sqcup y = \top^{\prod \mathfrak{A}}) &\Leftrightarrow \\ \exists y \in \prod \mathfrak{A} \forall i \in \operatorname{dom} \mathfrak{A} : (x_i \sqcap y_i = \bot^{\mathfrak{A}_i} \wedge x_i \sqcup y_i = \top^{\mathfrak{A}_i}) &\Leftrightarrow \\ \forall i \in \operatorname{dom} \mathfrak{A} \exists y \in \mathfrak{A}_i : (x_i \sqcap y = \bot^{\mathfrak{A}_i} \wedge x_i \sqcup y = \top^{\mathfrak{A}_i}) &\Leftrightarrow \\ \forall i \in \operatorname{dom} \mathfrak{A} : x_i \in Z(\mathfrak{A}_i). & \end{aligned}$$

So

$$Z\Big(\prod \mathfrak{A}\Big) = \prod \mathfrak{Z} \Leftrightarrow \prod_{i \in \operatorname{dom} \mathfrak{A}} Z(\mathfrak{A}_i) = \prod \mathfrak{Z} \Leftrightarrow$$
$$\text{(because every } \mathfrak{Z}_i \text{ is nonempty)} \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A} : Z(\mathfrak{A}_i) = \mathfrak{Z}_i.$$

□

Proposition 653. For every element a of a product filtrator $(\prod \mathfrak{A}, \prod \mathfrak{Z})$:

1°. $\operatorname{up} a = \prod_{i \in \operatorname{dom} a} \operatorname{up} a_i$;
2°. $\operatorname{down} a = \prod_{i \in \operatorname{dom} a} \operatorname{down} a_i$.

Proof. We will prove only the first as the second is dual.

$$\operatorname{up} a = \left\{ \frac{c \in \prod \mathfrak{Z}}{c \sqsupseteq a} \right\} = \left\{ \frac{c \in \prod \mathfrak{Z}}{\forall i \in \operatorname{dom} a : c_i \sqsupseteq a_i} \right\} =$$
$$\left\{ \frac{c \in \prod \mathfrak{Z}}{\forall i \in \operatorname{dom} a : c_i \in \operatorname{up} a_i} \right\} = \prod_{i \in \operatorname{dom} a} \operatorname{up} a_i.$$

□

Proposition 654. If every $(\mathfrak{A}_{i \in n}, \mathfrak{Z}_{i \in n})$ is a prefiltered filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a prefiltered filtrator.

Proof. Let $a, b \in \prod \mathfrak{A}$ and $a \neq b$. Then there exists $i \in n$ such that $a_i \neq b_i$ and so $\operatorname{up} a_i \neq \operatorname{up} b_i$. Consequently $\prod_{i \in \operatorname{dom} a} \operatorname{up} a_i \neq \prod_{i \in \operatorname{dom} a} \operatorname{up} b_i$ that is $\operatorname{up} a \neq \operatorname{up} b$. □

Proposition 655. Let every $(\mathfrak{A}_{i \in n}, \mathfrak{Z}_{i \in n})$ be a filtered filtrator with $\operatorname{up} x \neq \emptyset$ for every $x \in \mathfrak{A}_i$ (for every $i \in n$). Then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a filtered filtrator.

Proof. Let every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be a filtered filtrator. Let $\operatorname{up} a \supseteq \operatorname{up} b$ for some $a, b \in \prod \mathfrak{A}$. Then $\prod_{i \in \operatorname{dom} a} \operatorname{up} a_i \supseteq \prod_{i \in \operatorname{dom} a} \operatorname{up} b_i$ and consequently (taking into account that $\operatorname{up} x \neq \emptyset$ for every $x \in \mathfrak{A}_i$) $\operatorname{up} a_i \supseteq \operatorname{up} b_i$ for every $i \in n$. Then $\forall i \in n : a_i \sqsubseteq b_i$ that is $a \sqsubseteq b$. □

Proposition 656. Let $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be filtrators and each $\mathfrak{Z}_i$ be a complete lattice with $\operatorname{up} x \neq \emptyset$ for every $x \in \mathfrak{A}_i$ (for every $i \in n$). For $a \in \prod \mathfrak{A}$:

1°. $\operatorname{Cor} a = \lambda i \in \operatorname{dom} a : \operatorname{Cor} a_i$;
2°. $\operatorname{Cor}' a = \lambda i \in \operatorname{dom} a : \operatorname{Cor}' a_i$.

Proof. We will prove only the first, because the second is dual.

$$\operatorname{Cor} a =$$
$$\prod \mathfrak{Z} \bigsqcap \operatorname{up} a =$$
$$\lambda i \in \operatorname{dom} a : \bigsqcap^{\mathfrak{Z}_i} \left\{ \frac{x_i}{x \in \operatorname{up} a} \right\} = (\operatorname{up} x \neq \emptyset \text{ taken into account})$$
$$\lambda i \in \operatorname{dom} a : \bigsqcap^{\mathfrak{Z}_i} \left\{ \frac{x}{x \in \operatorname{up} a_i} \right\} =$$
$$\lambda i \in \operatorname{dom} a : \bigsqcap^{\mathfrak{Z}_i} \operatorname{up} a_i =$$
$$\lambda i \in \operatorname{dom} a : \operatorname{Cor} a_i.$$

□

Proposition 657. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a filtrator with (co)separable core and each $\mathfrak{A}_i$ has a least (greatest) element, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a filtrator with (co)separable core.

Proof. We will prove only for separable core, as co-separable core is dual.

$$x \asymp^{\prod \mathfrak{A}} y \Leftrightarrow$$
$$\text{(used the fact that } \mathfrak{A}_i \text{ has a least element)}$$
$$\forall i \in \operatorname{dom} \mathfrak{A} : x_i \asymp^{\mathfrak{A}_i} y_i \Rightarrow$$
$$\forall i \in \operatorname{dom} \mathfrak{A} \exists X \in \operatorname{up} x_i : X \asymp^{\mathfrak{A}_i} y_i \Leftrightarrow$$
$$\exists X \in \operatorname{up} x \forall i \in \operatorname{dom} \mathfrak{A} : X_i \asymp^{\mathfrak{A}_i} y_i \Leftrightarrow$$
$$\exists X \in \operatorname{up} x : X \asymp^{\prod \mathfrak{A}} y$$

for every $x, y \in \prod \mathfrak{A}$. □

Obvious 658.

1°. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a down-aligned filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a down-aligned filtrator.
2°. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is an up-aligned filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is an up-aligned filtrator.

Obvious 659.

1°. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a weakly down-aligned filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a weakly down-aligned filtrator.
2°. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a weakly up-aligned filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a weakly up-aligned filtrator.

Proposition 660. If every b_i is substractive from a_i where a and b are n-indexed families of elements of distributive lattices with least elements (where n is an index set), then $a \setminus b = \lambda i \in n : a_i \setminus b_i$.

Proof. We need to prove $(\lambda i \in n : a_i \setminus b_i) \sqcap b = \bot$ and $a \sqcup b = b \sqcup (\lambda i \in n : a_i \setminus b_i)$. Really

$$(\lambda i \in n : a_i \setminus b_i) \sqcap b = \lambda i \in n : (a_i \setminus b_i) \sqcap b_i = \bot;$$
$$b \sqcup (\lambda i \in n : a_i \setminus b_i) = \lambda i \in n : b_i \sqcup (a_i \setminus b_i) = \lambda i \in n : b_i \sqcup a_i = a \sqcup b.$$

□

PROPOSITION 661. If every $\mathfrak{A}_i$ is a distributive lattice, then $a \setminus^* b = \lambda i \in \operatorname{dom}\mathfrak{A} : a_i \setminus^* b_i$ for every $a, b \in \prod \mathfrak{A}$ whenever every $a_i \setminus^* b_i$ is defined.

PROOF. We need to prove that $\lambda i \in \operatorname{dom}\mathfrak{A} : a_i \setminus^* b_i = \bigsqcap\left\{\frac{z\in\prod\mathfrak{A}}{a\sqsubseteq b\sqcup z}\right\}$.

To prove it is enough to show $a_i \setminus^* b_i = \bigsqcap\left\{\frac{z_i}{z\in\prod\mathfrak{A}, a\sqsubseteq b\sqcup z}\right\}$ that is $a_i \setminus^* b_i = \bigsqcap\left\{\frac{z\in\mathfrak{A}_i}{a_i\sqsubseteq b_i\sqcup z}\right\}$ because $z' \in \left\{\frac{z_i}{z\in\prod\mathfrak{A}, a\sqsubseteq b\sqcup z}\right\} \Leftrightarrow z' \in \left\{\frac{z\in\mathfrak{A}_i}{a_i\sqsubseteq b_i\sqcup z}\right\}$ (for the reverse implication take $z_j = a_i$ for $j \neq i$), but $a_i \setminus^* b_i = \bigsqcap\left\{\frac{z\in\mathfrak{A}_i}{a_i\sqsubseteq b_i\sqcup z}\right\}$ is true by definition. □

PROPOSITION 662. If every $\mathfrak{A}_i$ is a distributive lattice with least element, then $a \# b = \lambda i \in \operatorname{dom}\mathfrak{A} : a_i \# b_i$ for every $a, b \in \prod \mathfrak{A}$ whenever every $a_i \# b_i$ is defined.

PROOF. We need to prove that $\lambda i \in \operatorname{dom}\mathfrak{A} : a_i \# b_i = \bigsqcup\left\{\frac{z\in\prod\mathfrak{A}}{z\sqsubseteq a\wedge z\asymp b}\right\}$.

To prove it is enough to show $a_i \# b_i = \bigsqcup\left\{\frac{z_i}{z\in\prod\mathfrak{A}, z\sqsubseteq a\wedge z\asymp b}\right\}$ that is $a_i \# b_i = \bigsqcup\left\{\frac{z_i}{z\in\prod\mathfrak{A}, z_i\sqsubseteq a_i\wedge\forall j\in\operatorname{dom}\mathfrak{A}: z_j\asymp b_j}\right\}$ that is $a_i \# b_i = \bigsqcup\left\{\frac{z\in\mathfrak{A}_i}{z\sqsubseteq a_i\wedge z\asymp b_i}\right\}$ (take $z_j = \bot^{\mathfrak{A}_j}$ for $j \neq i$) what is true by definition. □

PROPOSITION 663. Let every $\mathfrak{A}_i$ be a poset with least element and a_i^* is defined. Then $a^* = \lambda i \in \operatorname{dom}\mathfrak{A} : a_i^*$.

PROOF. We need to prove that $\lambda i \in \operatorname{dom}\mathfrak{A} : a_i^* = \bigsqcup\left\{\frac{c\in\prod\mathfrak{A}}{c\asymp a}\right\}$. To prove this it is enough to show that $a_i^* = \bigsqcup\left\{\frac{c_i}{c\in\prod\mathfrak{A}, c\asymp a}\right\}$ that is $a_i^* = \bigsqcup\left\{\frac{c_i}{c\in\prod\mathfrak{A}, \forall j\in\operatorname{dom}\mathfrak{A}: c_j\asymp a_j}\right\}$ that is $a_i^* = \bigsqcup\left\{\frac{c_i}{c\in\prod\mathfrak{A}, c_i\asymp a_i}\right\}$ (take $c_j = \bot^{\mathfrak{A}_j}$ for $j \neq i$) that is $a_i^* = \bigsqcup\left\{\frac{c\in\mathfrak{A}_i}{c\asymp a_i}\right\}$ what is true by definition. □

COROLLARY 664. Let every $\mathfrak{A}_i$ be a poset with greatest element and a_i^+ is defined. Then $a^+ = \lambda i \in \operatorname{dom}\mathfrak{A} : a_i^+$.

PROOF. By duality. □

5.34. Filters on a Set

In this section we will fix a powerset filtrator $(\mathfrak{A}, \mathfrak{Z}) = (\mathfrak{A}, \mathscr{P}\mathfrak{U})$ for some set $\mathfrak{U}$.

The consideration below is about filters on a set $\mathfrak{U}$, but this can be generalized for filters on complete atomic boolean algebras due complete atomic boolean algebras are isomorphic to algebras of sets on some set $\mathfrak{U}$.

5.34.1. Fréchet Filter.

DEFINITION 665. $\Omega = \left\{\frac{\mathfrak{U}\setminus X}{X \text{ is a finite subset of } \mathfrak{U}}\right\}$ is called either *Fréchet filter* or *cofinite filter*.

It is trivial that Fréchet filter is a filter.

PROPOSITION 666. $\operatorname{Cor}\Omega = \bot^{\mathfrak{Z}}$; $\bigcap\Omega = \emptyset$.

PROOF. This can be deduced from the formula $\forall\alpha \in \mathfrak{U}\exists X \in \Omega : \alpha \notin X$. □

THEOREM 667. $\max\left\{\frac{\mathcal{X}\in\mathfrak{A}}{\operatorname{Cor}\mathcal{X}=\bot^{\mathfrak{Z}}}\right\} = \max\left\{\frac{\mathcal{X}\in\mathfrak{A}}{\bigcap\mathcal{X}=\emptyset}\right\} = \Omega$.

PROOF. Due the last proposition, it is enough to show that $\operatorname{Cor}\mathcal{X} = \bot^3 \Rightarrow \mathcal{X} \sqsubseteq \Omega$ for every filter $\mathcal{X}$.

Let $\operatorname{Cor}\mathcal{X} = \bot^3$ for some filter $\mathcal{X}$. Let $X \in \Omega$. We need to prove that $X \in \mathcal{X}$.

$X = \mathfrak{U} \setminus \{\alpha_0, \dots, \alpha_n\}$. $\mathfrak{U} \setminus \{\alpha_i\} \in \mathcal{X}$ because otherwise $\alpha_i \in \uparrow^{-1} \operatorname{Cor}\mathcal{X}$. So $X \in \mathcal{X}$. □

THEOREM 668. $\Omega = \bigsqcup^{\mathfrak{A}} \{ \frac{x}{x \text{ is a non-trivial ultrafilter}} \}$.

PROOF. It follows from the facts that $\operatorname{Cor} x = \bot^3$ for every non-trivial ultrafilter x, that $\mathfrak{A}$ is an atomistic lattice, and the previous theorem. □

THEOREM 669. Cor is the lower adjoint of $\Omega \sqcup^{\mathfrak{A}} -$.

PROOF. Because both Cor and $\Omega \sqcup^{\mathfrak{A}} -$ are monotone, it is enough (theorem 126) to prove (for every filters $\mathcal{X}$ and $\mathcal{Y}$)

$$\mathcal{X} \sqsubseteq \Omega \sqcup^{\mathfrak{A}} \operatorname{Cor}\mathcal{X} \quad \text{and} \quad \operatorname{Cor}(\Omega \sqcup^{\mathfrak{A}} \mathcal{Y}) \sqsubseteq \mathcal{Y}.$$

$\operatorname{Cor}(\Omega \sqcup^{\mathfrak{A}} \mathcal{Y}) = \operatorname{Cor}\Omega \sqcup^3 \operatorname{Cor}\mathcal{Y} = \bot^3 \sqcup^3 \operatorname{Cor}\mathcal{Y} = \operatorname{Cor}\mathcal{Y} \sqsubseteq \mathcal{Y}$.

$\Omega \sqcup^{\mathfrak{A}} \operatorname{Cor}\mathcal{X} \sqsupseteq \operatorname{Edg}\mathcal{X} \sqcup^{\mathfrak{A}} \operatorname{Cor}\mathcal{X} = \mathcal{X}$. □

COROLLARY 670. $\operatorname{Cor}\mathcal{X} = \mathcal{X} \setminus^* \Omega$ for every filter on a set.

PROOF. By theorem 154. □

COROLLARY 671. $\operatorname{Cor} \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{A}} \langle \operatorname{Cor} \rangle^* S$ for any set S of filters on a powerset.

This corollary can be rewritten in elementary terms and proved elementarily:

PROPOSITION 672. $\bigcap \bigcap S = \bigcup_{F \in S} \bigcap F$ for a set S of filters on some set.

PROOF. (by ANDREAS BLASS) The $\supseteq$ direction is rather formal. Consider any one of the sets being intersected on the left side, i.e., any set X that is in all the filters in S, and consider any of the sets being unioned (that's not a word, but you know what I mean) on the right, i.e., $\bigcap F$ for some $F \in S$. Then, since $X \in F$, we have $\bigcap F \subseteq X$. Taking the union over all $F \in S$ (while keeping X fixed), we get that the right side of your equation is $\subseteq X$. Since that's true for all $X \in \bigcap S$, we infer that the right side is a subset of the left side. (This argument seems to work in much greater generality; you just need that the relevant infima (in place of intersections) exist in your poset.)

For the $\subseteq$ direction, consider any element $x \in \bigcap \bigcap S$, and suppose, toward a contradiction, that it is not an element of the union on the right side of your equation. So, for each $F \in S$, we have $x \notin \bigcap F$, and therefore we can find a set $A_F \in F$ with $x \notin A_F$. Let $B = \bigcup_{F \in S} A_F$ and notice that $B \in F$ for every $F \in S$ (because $B \supseteq A_F$). So $B \in \bigcap S$. But, by choice of the A_F's, we have $x \notin B$, contrary to the assumption that $x \in \bigcap \bigcap S$. □

PROPOSITION 673. $\partial\Omega(U)$ is the set of infinite subsets of U.

PROOF. $\partial\Omega(U) = \neg\langle\neg\rangle^*\Omega(U)$.

$\langle\neg\rangle^*\Omega$ is the set of finite subsets of U. Thus $\neg\langle\neg\rangle^*\Omega(U)$ is the set of infinite subsets of U. □

5.34.2. Number of Filters on a Set.

DEFINITION 674. A collection Y of sets has finite intersection property iff intersection of any finite subcollection of Y is non-empty.

The following was borrowed from [7]. Thanks to ANDREAS BLASS for email support about his proof.

LEMMA 675. (by HAUSDORFF) For an infinite set X there is a family $\mathcal{F}$ of $2^{\operatorname{card} X}$ many subsets of X such that given any disjoint finite subfamilies $\mathcal{A}$, $\mathcal{B}$, the intersection of sets in $\mathcal{A}$ and complements of sets in $\mathcal{B}$ is nonempty.

PROOF. Let

$$X' = \left\{ \frac{(P,Q)}{P \in \mathscr{P}X \text{ is finite}, Q \in \mathscr{P}\mathscr{P}P} \right\}.$$

It's easy to show that $\operatorname{card} X' = \operatorname{card} X$. So it is enough to show this for X' instead of X. Let

$$\mathcal{F} = \left\{ \frac{\left\{ \frac{(P,Q) \in X'}{Y \cap P \in Q} \right\}}{Y \in \mathscr{P}X} \right\}.$$

To finish the proof we show that for every disjoint finite $Y_+ \in \mathscr{P}\mathscr{P}X$ and finite $Y_- \in \mathscr{P}\mathscr{P}X$ there exist $(P,Q) \in X'$ such that

$$\forall Y \in Y_+ : (P,Q) \in \left\{ \frac{(P,Q) \in X'}{Y \cap P \in Q} \right\} \quad \text{and} \quad \forall Y \in Y_- : (P,Q) \notin \left\{ \frac{(P,Q) \in X'}{Y \cap P \in Q} \right\}$$

what is equivalent to existence $(P,Q) \in X'$ such that

$$\forall Y \in Y_+ : Y \cap P \in Q \quad \text{and} \quad \forall Y \in Y_- : Y \cap P \notin Q.$$

For existence of this (P,Q), it is enough existence of P such that intersections $Y \cap P$ are different for different $Y \in Y_+ \cup Y_-$.

Really, for each pair of distinct $Y_0, Y_1 \in Y_+ \cup Y_-$ choose a point which lies in one of the sets Y_0, Y_1 and not in an other, and call the set of such points P. Then $Y \cap P$ are different for different $Y \in Y_+ \cup Y_-$. □

COROLLARY 676. For an infinite set X there is a family $\mathcal{F}$ of $2^{\operatorname{card} X}$ many subsets of X such that for arbitrary disjoint subfamilies $\mathcal{A}$ and $\mathcal{B}$ the set $\mathcal{A} \cup \left\{ \frac{X \setminus A}{A \in \mathcal{B}} \right\}$ has finite intersection property.

THEOREM 677. Let X be a set. The number of ultrafilters on X is $2^{2^{\operatorname{card} X}}$ if X is infinite and $\operatorname{card} X$ if X is finite.

PROOF. The finite case follows from the fact that every ultrafilter on a finite set is trivial. Let X be infinite. From the lemma, there exists a family $\mathcal{F}$ of $2^{\operatorname{card} X}$ many subsets of X such that for every $\mathcal{G} \in \mathscr{P}\mathcal{F}$ we have $\Phi(\mathcal{F},\mathcal{G}) = \bigsqcap^{\mathfrak{A}} \mathcal{G} \sqcap \bigsqcap^{\mathfrak{A}} \left\{ \frac{X \setminus A}{A \in \mathcal{F} \setminus \mathcal{G}} \right\} \neq \perp^{\mathfrak{A}(X)}$.

This filter contains all sets from $\mathcal{G}$ and does not contain any sets from $\mathcal{F} \setminus \mathcal{G}$. So for every suitable pairs $(\mathcal{F}_0, \mathcal{G}_0)$ and $(\mathcal{F}_1, \mathcal{G}_1)$ there is $A \in \Phi(\mathcal{F}_0, \mathcal{G}_0)$ such that $\overline{A} \in \Phi(\mathcal{F}_1, \mathcal{G}_1)$. Consequently all filters $\Phi(\mathcal{F},\mathcal{G})$ are disjoint. So for every pair $(\mathcal{F},\mathcal{G})$ where $\mathcal{G} \in \mathscr{P}\mathcal{F}$ there exist a distinct ultrafilter under $\Phi(\mathcal{F},\mathcal{G})$, but the number of such pairs $(\mathcal{F},\mathcal{G})$ is $2^{2^{\operatorname{card} X}}$. Obviously the number of all filters is not above $2^{2^{\operatorname{card} X}}$. □

COROLLARY 678. The number of filters on $\mathfrak{U}$ is $2^{2^{\operatorname{card} X}}$ if $\mathfrak{U}$ us infinite and $2^{\operatorname{card} \mathfrak{U}}$ if $\mathfrak{U}$ is finite.

PROOF. The finite case is obvious. The infinite case follows from the theorem and the fact that filters are collections of sets and there cannot be more than $2^{2^{\operatorname{card} \mathfrak{U}}}$ collections of sets on $\mathfrak{U}$. □

5.35. Bases on filtrators

DEFINITION 679. A set S of binary relations is a *base* on a filtrator $(\mathfrak{A}, \mathfrak{Z})$ of $f \in \mathfrak{A}$ when all elements of S are above f and $\forall X \in \operatorname{up} f \exists T \in S : T \sqsubseteq X$.

OBVIOUS 680. Every base on an up-aligned filtrator is nonempty.

PROPOSITION 681. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered filtrator.
4°. A set $S \in \mathscr{P}\mathfrak{Z}$ is a base of a filtrator element iff $\bigsqcap^{\mathfrak{A}} S$ exists and S is a base of $\bigsqcap^{\mathfrak{A}} S$.

PROOF.

1°⇒2°, 2°⇒3°. Obvious.

3°⇒4°.

⇐. Obvious.

⇒. Let S be a base of an $f \in \mathfrak{A}$. f is obviously a lower bound of S. Let g be a lower bound of S. Then for every $X \in \operatorname{up} f$ we have $g \sqsubseteq X$ that is $X \in \operatorname{up} g$. Thus $\operatorname{up} f \subseteq \operatorname{up} g$ and thus $f \sqsupseteq g$ that is f is the greatest upper bound of S.

□

PROPOSITION 682. There exists an $f \in \mathfrak{A}$ such that $\operatorname{up} f = S$ iff S is a base and is an upper set (for every set $S \in \mathscr{P}\mathfrak{Z}$).

PROOF.

⇒. If $\operatorname{up} f = S$ then S is an upper set and S is a base of f because $\forall X \in \operatorname{up} f \exists T \in S : T = X$.

⇐. Let S be a base of some filtrator element f and is an upper set. Then for every $X \in \operatorname{up} f$ there is $T \in S$ such that $T \sqsubseteq X$. Thus $X \in S$. We have $\operatorname{up} f \subseteq S$. But $S \subseteq \operatorname{up} f$ is obvious. We have $\operatorname{up} f = S$.

□

PROPOSITION 683. $\operatorname{up} f$ is a base of f for every $f \in \mathfrak{A}$.

PROOF. Denote $S = \operatorname{up} f$. That f is a lower bound of S is obvious.
If $X \in \operatorname{up} f$ then $\exists T \in S : T = X$. Thus S is a base of f. □

PROPOSITION 684. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered filtrator.
4°. $f = \bigsqcap^{\mathfrak{A}} S$ for every base S of an $f \in \mathfrak{A}$.

PROOF.

1°⇒2°, 2°⇒3°. Obvious.

3°⇒4°. f is a lower bound of S by definition.

Let g be a lower bound of S. Then for every $X \in \operatorname{up} f$ there we have $g \sqsubseteq X$ that is $X \in \operatorname{up} g$. Thus $\operatorname{up} f \subseteq \operatorname{up} g$ and thus $f \sqsupseteq g$ that is f is the greatest lower bound of S.

□

PROPOSITION 685. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered filtrator.
4°. If S is a base on a filtrator, then $\bigsqcap^{\mathfrak{A}} S$ exists and $\operatorname{up} \bigsqcap^{\mathfrak{A}} S = \bigcup_{K \in S} \operatorname{up} K$.

PROOF.
1°⇒2°, 2°⇒3°. Obvious.
3°⇒4°. $\bigsqcap^{\mathfrak{A}} S$ exists because our filtrator is filtered. Above we proved that S is a base of $\bigsqcap^{\mathfrak{A}} S$. That $\bigcup_{K \in S} \operatorname{up} K \subseteq \operatorname{up} \bigsqcap^{\mathfrak{A}} S$ is obvious. If $X \in \operatorname{up} \bigsqcap^{\mathfrak{A}} S$ then by properties of bases we have $K \in S$ such that $K \sqsubseteq X$. Thus $X \in \operatorname{up} K$ and so $X \in \bigcup_{K \in S} \operatorname{up} K$. So $\operatorname{up} \bigsqcap^{\mathfrak{A}} S \subseteq \bigcup_{K \in S} \operatorname{up} K$.

□

PROPOSITION 686. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with binarily meet-closed core such that $\forall a \in \mathfrak{A} : \operatorname{up} a \neq \emptyset$.
4°. A base on the filtrator $(\mathfrak{A}; \mathfrak{Z})$ is the same as base of a filter (on $\mathfrak{Z}$).

PROOF.
1°⇒2°. Obvious.
2°⇒3°. Corollary 536.
3°⇒4°.

⇒. Let S be a base of f on the filtrator $(\mathfrak{A}; \mathfrak{Z})$. Then for every $a, b \in S$ we have $a, b \in \operatorname{up} f$ and thus $a \sqcap^{\mathfrak{Z}} b = a \sqcap^{\mathfrak{A}} b \in \operatorname{up} f$. Thus $\exists x \in S : x \sqsubseteq a \sqcap^{\mathfrak{Z}} b$ that is $x \sqsubseteq a \wedge x \sqsubseteq b$. It remains to show that S is nonempty, but this follows from $\operatorname{up} a$ being nonempty.

⇐. Let S be a base of filter f (on $\mathfrak{Z}$). Let $X \in \operatorname{up} f$. Then there is $T \in S$ such that $T \sqsubseteq X$.

□

5.36. Some Counter-Examples

EXAMPLE 687. There exist a bounded distributive lattice which is not lattice with separable center.

PROOF. The lattice with the Hasse diagram[2] on figure 5 is bounded and distributive because it does not contain "diamond lattice" nor "pentagon lattice" as a sublattice [46].

It's center is $\{0, 1\}$. $x \sqcap y = 0$ despite $\operatorname{up} x = \{x, a, 1\}$ but $y \sqcap 1 \neq 0$ consequently the lattice is not with separable center. □

In this section $\mathfrak{A}$ denotes the set of filters on a set.

EXAMPLE 688. There is a separable poset (that is a set with $\star$ being an injection) which is not strongly separable (that is $\star$ isn't order reflective).

PROOF. (with help of sci.math partakers) Consider a poset with the Hasse diagram 6.

Then $\star p = \{p, a, b\}$, $\star q = \{q, a, b\}$, $\star r = \{r, b\}$, $\star a = \{p, q, a, b\}$, $\star b = \{p, q, a, b, r\}$.

Thus $\star x = \star y \Rightarrow x = y$ for any x, y in our poset.

$\star a \subseteq \star b$ but not $a \sqsubseteq b$. □

EXAMPLE 689. There is a prefiltered filtrator which is not filtered.

[2]See Wikipedia for a definition of Hasse diagrams.

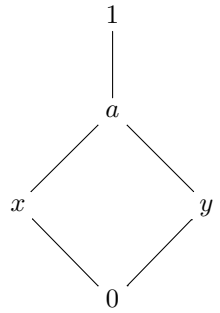

FIGURE 5

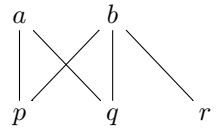

FIGURE 6

PROOF. (MATTHIAS KLUPSCH) Take $\mathfrak{A} = \{a, b\}$ with the order being equality and $\mathfrak{Z} = \{b\}$. Then $\operatorname{up} a = \emptyset \sqsubseteq \{b\} = \operatorname{up} b$, so up is injective, hence the filtrator is prefiltered, but because of $a \not\sqsubseteq b$ the filtrator is not filtered. □

For further examples we will use the filter Δ defined by the formula

$$\Delta = \bigsqcap^{\mathfrak{A}} \left\{ \frac{]-\epsilon;\epsilon[}{\epsilon \in \mathbb{R}, \epsilon > 0} \right\}$$

and more general

$$\Delta + a = \bigsqcap^{\mathfrak{A}} \left\{ \frac{]a-\epsilon;a+\epsilon[}{\epsilon \in \mathbb{R}, \epsilon > 0} \right\}.$$

EXAMPLE 690. There exists $A \in \mathscr{P}U$ such that $\bigsqcap^{\mathfrak{A}} A \neq \bigsqcap A$.

PROOF. $\bigsqcap^{\mathfrak{Z}} \left\{ \frac{]-\epsilon;\epsilon[}{\epsilon \in \mathbb{R}, \epsilon > 0} \right\} = \uparrow \{0\} \neq \Delta$. □

EXAMPLE 691. There exists a set U and a filter a and a set S of filters on the set U such that $a \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \bigsqcup^{\mathfrak{A}} \langle a \sqcap^{\mathfrak{A}} \rangle^* S$.

PROOF. Let $a = \Delta$ and $S = \left\{ \frac{\uparrow^{\mathbb{R}}]\epsilon;+\infty[}{\epsilon > 0} \right\}$. Then $a \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S = \Delta \sqcap^{\mathfrak{A}}]0;+\infty[) \neq \bot^{\mathfrak{A}}$ while $\bigsqcup^{\mathfrak{A}} \langle a \sqcap^{\mathfrak{A}} \rangle^* S = \bigsqcup^{\mathfrak{A}} \{\bot^{\mathfrak{A}}\} = \bot^{\mathfrak{A}}$. □

EXAMPLE 692. There are tornings which are not weak partitions.

PROOF. $\left\{ \frac{\Delta + a}{a \in \mathbb{R}} \right\}$ is a torning but not weak partition of the real line. □

LEMMA 693. Let $\mathfrak{A}$ be the set of filters on a set U. Then $X \sqcap^{\mathfrak{A}} \Omega \sqsubseteq Y \sqcap^{\mathfrak{A}} \Omega$ iff $X \setminus Y$ is a finite set, having fixed sets $X, Y \in \mathscr{P}U$.

PROOF. Let M be the set of finite subsets of U.

$$\begin{aligned} X \sqcap^{\mathfrak{A}} \Omega \sqsubseteq Y \sqcap^{\mathfrak{A}} \Omega &\Leftrightarrow \\ \left\{ \frac{X \cap K_X}{K_X \in \Omega} \right\} \supseteq \left\{ \frac{Y \cap K_Y}{K_Y \in \Omega} \right\} &\Leftrightarrow \\ \forall K_Y \in \Omega \exists K_X \in \Omega : Y \cap K_Y = X \cap K_X &\Leftrightarrow \\ \forall L_Y \in M \exists L_X \in M : Y \setminus L_Y = X \setminus L_X &\Leftrightarrow \\ \forall L_Y \in M : X \setminus (Y \setminus L_Y) \in M &\Leftrightarrow \\ X \setminus Y \in M. \end{aligned}$$

□

EXAMPLE 694. There exists a filter $\mathcal{A}$ on a set U such that $(\mathscr{P}U)/{\sim}$ and $Z(D\mathcal{A})$ are not complete lattices.

PROOF. Due to the isomorphism it is enough to prove for $(\mathscr{P}U)/{\sim}$.

Let take $U = \mathbb{N}$ and $\mathcal{A} = \Omega$ be the Fréchet filter on $\mathbb{N}$.

Partition $\mathbb{N}$ into infinitely many infinite sets $A_0, A_1, \dots$. To withhold our example we will prove that the set $\{[A_0], [A_1], \dots\}$ has no supremum in $(\mathscr{P}U)/{\sim}$.

Let $[X]$ be an upper bound of $[A_0], [A_1], \dots$ that is $\forall i \in \mathbb{N} : X \sqcap^{\mathfrak{A}} \Omega \sqsupseteq A_i \sqcap^{\mathfrak{A}} \Omega$ that is $A_i \setminus X$ is finite. Consequently X is infinite. So $X \cap A_i \neq \emptyset$.

Choose for every $i \in \mathbb{N}$ some $z_i \in X \cap A_i$. The $\{z_0, z_1, \dots\}$ is an infinite subset of X (take into account that $z_i \neq z_j$ for $i \neq j$). Let $Y = X \setminus \{z_0, z_1, \dots\}$. Then $Y \sqcap^{\mathfrak{A}} \Omega \sqsupseteq A_i \sqcap^{\mathfrak{A}} \Omega$ because $A_i \setminus Y = A_i \setminus (X \setminus \{z_i\}) = (A_i \setminus X) \cup \{z_i\}$ which is finite because $A_i \setminus X$ is finite. Thus $[Y]$ is an upper bound for $\{[A_0], [A_1], \dots\}$.

Suppose $Y \sqcap^{\mathfrak{A}} \Omega = X \sqcap^{\mathfrak{A}} \Omega$. Then $Y \setminus X$ is finite what is not true. So $Y \sqcap^{\mathfrak{A}} \Omega \sqsubset X \sqcap^{\mathfrak{A}} \Omega$ that is $[Y]$ is below $[X]$. □

5.36.1. Weak and Strong Partition.

DEFINITION 695. A family S of subsets of a countable set is *independent* iff the intersection of any finitely many members of S and the complements of any other finitely many members of S is infinite.

LEMMA 696. The "infinite" at the end of the definition could be equivalently replaced with "nonempty" if we assume that S is infinite.

PROOF. Suppose that some sets from the above definition has a finite intersection J of cardinality n. Then (thanks S is infinite) get one more set $X \in S$ and we have $J \cap X \neq \emptyset$ and $J \cap (\mathbb{N} \setminus X) \neq \emptyset$. So $\operatorname{card}(J \cap X) < n$. Repeating this, we prove that for some finite family of sets we have empty intersection what is a contradiction. □

LEMMA 697. There exists an independent family on $\mathbb{N}$ of cardinality $\mathfrak{c}$.

PROOF. Let C be the set of finite subsets of $\mathbb{Q}$. Since $\operatorname{card} C = \operatorname{card} \mathbb{N}$, it suffices to find $\mathfrak{c}$ independent subsets of C. For each $r \in \mathbb{R}$ let

$$E_r = \left\{ \frac{F \in C}{\operatorname{card}(F \cap]-\infty; r[) \text{ is even}} \right\}.$$

All E_{r_1} and E_{r_2} are distinct for distinct $r_1, r_2 \in \mathbb{R}$ since we may consider $F = \{r'\} \in C$ where a rational number r' is between r_1 and r_2 and thus F is a member of exactly one of the sets E_{r_1} and E_{r_2}. Thus $\operatorname{card}\left\{ \frac{E_r}{r \in \mathbb{R}} \right\} = \mathfrak{c}$.

We will show that $\left\{ \frac{E_r}{r \in \mathbb{R}} \right\}$ is independent. Let $r_1, \dots, r_k, s_1, \dots, s_k$ be distinct reals. It is enough to show that these have a nonempty intersection, that is existence of some F such that F belongs to all the E_r and none of E_s.

But this can be easily accomplished taking F having zero or one element in each of intervals to which $r_1, \dots, r_k, s_1, \dots, s_k$ split the real line. $\square$

EXAMPLE 698. There exists a weak partition of a filter on a set which is not a strong partition.

PROOF. (suggested by ANDREAS BLASS) Let $\left\{\frac{X_r}{r\in\mathbb{R}}\right\}$ be an independent family of subsets of $\mathbb{N}$. We can assume $a \neq b \Rightarrow X_a \neq X_b$ due the above lemma.

Let $\mathcal{F}_a$ be a filter generated by X_a and the complements $\mathbb{N} \setminus X_b$ for all $b \in \mathbb{R}$, $b \neq a$. Independence implies that $\mathcal{F}_a \neq \bot^{\mathfrak{A}}$ (by properties of filter bases).

Let $S = \left\{\frac{\mathcal{F}_r}{r\in\mathbb{R}}\right\}$. We will prove that S is a weak partition but not a strong partition.

Let $a \in \mathbb{R}$. Then $X_a \in \mathcal{F}_a$ while $\forall b \in \mathbb{R} \setminus \{a\} : \mathbb{N} \setminus X_a \in \mathcal{F}_b$ and therefore $\mathbb{N} \setminus X_a \in \bigsqcup^{\mathfrak{A}}\left\{\frac{\mathcal{F}_b}{\mathbb{R}\ni b\neq a}\right\}$. Therefore $\mathcal{F}_a \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}}\left\{\frac{\mathcal{F}_b}{\mathbb{R}\ni b\neq a}\right\} = \bot^{\mathfrak{A}}$. Thus S is a weak partition.

Suppose S is a strong partition. Then for each set $Z \in \mathscr{P}\mathbb{R}$

$$\bigsqcup^{\mathfrak{A}}\left\{\frac{\mathcal{F}_b}{b \in Z}\right\} \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}}\left\{\frac{\mathcal{F}_b}{b \in \mathbb{R} \setminus Z}\right\} = \bot^{\mathfrak{A}}$$

what is equivalent to existence of $M(Z) \in \mathscr{P}\mathbb{N}$ such that

$$M(Z) \in \bigsqcup^{\mathfrak{A}}\left\{\frac{\mathcal{F}_b}{b \in Z}\right\} \quad \text{and} \quad \mathbb{N} \setminus M(Z) \in \bigsqcup^{\mathfrak{A}}\left\{\frac{\mathcal{F}_b}{b \in \mathbb{R} \setminus Z}\right\}$$

that is

$$\forall b \in Z : M(Z) \in \mathcal{F}_b \quad \text{and} \quad \forall b \in \mathbb{R} \setminus Z : \mathbb{N} \setminus M(Z) \in \mathcal{F}_b.$$

Suppose $Z \neq Z' \in \mathscr{P}\mathbb{N}$. Without loss of generality we may assume that some $b \in Z$ but $b \notin Z'$. Then $M(Z) \in \mathcal{F}_b$ and $\mathbb{N} \setminus M(Z') \in \mathcal{F}_b$. If $M(Z) = M(Z')$ then $\mathcal{F}_b = \bot^{\mathfrak{A}}$ what contradicts to the above.

So M is an injective function from $\mathscr{P}\mathbb{R}$ to $\mathscr{P}\mathbb{N}$ what is impossible due cardinality issues. $\square$

LEMMA 699. (by NIELS DIEPEVEEN, with help of KARL KRONENFELD) Let K be a collection of nontrivial ultrafilters. We have $\bigsqcup K = \Omega$ iff $\exists \mathcal{G} \in K : A \in \operatorname{up}\mathcal{G}$ for every infinite set A.

PROOF.

$\Rightarrow$. Suppose $\bigsqcup K = \Omega$ and let A be a set such that $\nexists \mathcal{G} \in K : A \in \operatorname{up}\mathcal{G}$. Let's prove A is finite.

Really, $\forall \mathcal{G} \in K : \mathfrak{U} \setminus A \in \operatorname{up}\mathcal{G}$; $\mathfrak{U} \setminus A \in \operatorname{up}\Omega$; A is finite.

$\Leftarrow$. Let $\exists \mathcal{G} \in K : A \in \operatorname{up}\mathcal{G}$. Suppose A is a set in $\operatorname{up}\bigsqcup K$.

To finish the proof it's enough to show that $\mathfrak{U} \setminus A$ is finite.

Suppose $\mathfrak{U} \setminus A$ is infinite. Then $\exists \mathcal{G} \in K : \mathfrak{U} \setminus A \in \operatorname{up}\mathcal{G}$; $\exists \mathcal{G} \in K : A \notin \operatorname{up}\mathcal{G}$; $A \notin \operatorname{up}\bigsqcup K$, contradiction.

$\square$

LEMMA 700. (by NIELS DIEPEVEEN) If K is a non-empty set of ultrafilters such that $\bigsqcup K = \Omega$, then for every $\mathcal{G} \in K$ we have $\bigsqcup(K \setminus \{\mathcal{G}\}) = \Omega$.

PROOF. $\exists \mathcal{F} \in K : A \in \operatorname{up}\mathcal{F}$ for every infinite set A.

The set A can be partitioned into two infinite sets A_1, A_2.

Take $\mathcal{F}_1, \mathcal{F}_2 \in K$ such that $A_1 \in \mathcal{F}_1$, $A_2 \in \mathcal{F}_2$.

$\mathcal{F}_1 \neq \mathcal{F}_2$ because otherwise A_1 and A_2 are not disjoint.

Obviously $A \in \mathcal{F}_1$ and $A \in \mathcal{F}_2$.

So there exist two different $\mathcal{F} \in K$ such that $A \in \operatorname{up}\mathcal{F}$. Consequently $\exists \mathcal{F} \in K \setminus \{\mathcal{G}\} : A \in \operatorname{up}\mathcal{F}$ that is $\bigsqcup(K \setminus \{\mathcal{G}\}) = \Omega$. $\square$

EXAMPLE 701. There exists a filter on a set which cannot be weakly partitioned into ultrafilters.

PROOF. Consider cofinite filter Ω on any infinite set.

Suppose K is its weak partition into ultrafilters. Then $x \asymp \bigsqcup(K \setminus \{x\})$ for some ultrafilter $x \in K$.

We have $\bigsqcup(K \setminus \{x\}) \sqsubset \bigsqcup K$ (otherwise $x \sqsubseteq \bigsqcup(K \setminus \{x\})$) what is impossible due the last lemma. $\square$

COROLLARY 702. There exists a filter on a set which cannot be strongly partitioned into ultrafilters.

5.37. Open problems about filters

Under which conditions $a \setminus^* b$ and $a \# b$ are complementive to a?

Generalize straight maps for arbitrary posets.

5.38. Further notation

Below to define funcoids and reloids we need a fixed powerset filtrator.

Let $(\mathscr{F} A, \mathscr{T} A)$ be an arbitrary but fixed powerset filtrator. This filtrator exists by the theorem 462.

I will call elements of $\mathscr{F}$ *filter objects*.

For brevity we will denote lattice operations on $\mathscr{F} A$ without indexes (for example, take $\bigsqcap S = \bigsqcap^{\mathscr{F} A} S$ for $S \in \mathscr{P}\mathscr{F} A$).

Note that above we also took operations on $\mathscr{T} A$ without indexes (for example, take $\bigsqcap S = \bigsqcap^{\mathscr{T} A} S$ for $S \in \mathscr{P}\mathscr{T} A$).

Because we identify $\mathscr{T} A$ with principal elements of $\mathscr{F} A$, the notation like $\bigsqcap S$ for $S \in \mathscr{P}\mathscr{T} A$ would be inconsistent (it can mean both $\bigsqcap^{\mathscr{T} A} S$ or $\bigsqcap^{\mathscr{F} A} S$). We explicitly state that $\bigsqcap S$ in this case does *not* mean $\bigsqcap^{\mathscr{F} A} S$.

For $\mathcal{X} \in \mathscr{F}$ we will denote $\operatorname{GR} \mathcal{X}$ the corresponding filter on $\mathscr{P} A$. It is a convenient notation to describe relations between filters and sets, consider for example the formula: $\{x\} \subseteq \bigcap \operatorname{GR} \mathcal{X}$.

We will denote lattice operations without pointing a specific set like $\bigsqcap^{\mathscr{F}} S = \bigsqcap^{\mathscr{F}(A)} S$ for a set $S \in \mathscr{P}\mathscr{F}(A)$.

5.39. Equivalent filters and rebase of filters

Throughout this section we will assume that $\mathfrak{Z}$ is a lattice.

An important example: $\mathfrak{Z}$ is the lattice of all small (regarding some Grothendieck universe) sets. (This $\mathfrak{Z}$ is not a powerset, and even not a complete lattice.)

Throughout this section I will use the word *filter* to denote a filter on a sublattice DA where $A \in \mathfrak{Z}$ (if not told explicitly to be a filter on some other set).

The following is an embedding from filters $\mathcal{A}$ on a lattice DA into the lattice of filters on $\mathfrak{Z}$: $\mathscr{S}\mathcal{A} = \left\{ \frac{K \in \mathfrak{Z}}{\exists X \in \mathcal{A} : X \sqsubseteq K} \right\}$.

PROPOSITION 703. Values of this embedding are filters on the lattice $\mathfrak{Z}$.

PROOF. That $\mathscr{S}\mathcal{A}$ is an upper set is obvious.

Let $P, Q \in \mathscr{S}\mathcal{A}$. Then $P, Q \in \mathfrak{Z}$ and there is an $X \in \mathcal{A}$ such that $X \sqsubseteq P$ and $Y \in \mathcal{A}$ such that $Y \sqsubseteq Q$. So $X \sqcap Y \in \mathcal{A}$ and $P \sqcap Q \sqsupseteq X \sqcap Y \in \mathcal{A}$, so $P \sqcap Q \in \mathscr{S} A$. $\square$

5.39.1. Rebase of filters.

Definition 704. *Rebase* for every filter $\mathcal{A}$ and every $A \in \mathfrak{Z}$ is $\mathcal{A} \div A = \bigsqcap\left\{\frac{\uparrow^A(X\sqcap A)}{X\in\mathcal{A}}\right\}$.

Obvious 705. $\langle A\sqcap\rangle^*\mathscr{S}\mathcal{A}$ is a filter on A.

Proposition 706. The rebase conforms to the formula

$$\mathcal{A} \div A = \langle A\sqcap\rangle^*\mathscr{S}\mathcal{A}.$$

Proof. We know that $\langle A\sqcap\rangle^*\mathscr{S}\mathcal{A}$ is a filter.

If $P \in \langle A\sqcap\rangle^*\mathscr{S}\mathcal{A}$ then $P \in \mathscr{P}A$ and $Y \sqcap A \sqsubseteq P$ for some $Y \in \mathcal{A}$. Thus $P \sqsupseteq Y \sqcap A \in \bigsqcap\left\{\frac{\uparrow^A(Y\sqcap A)}{Y\in\mathcal{A}}\right\}$.

If $P \in \bigsqcap\left\{\frac{\uparrow^A(X\sqcap A)}{X\in\mathcal{A}}\right\}$ then by properties of generalized filter bases, there exists $X \in \mathcal{A}$ such that $P \sqsupseteq X \sqcap A$. Also $P \in \mathscr{P}A$. Thus $P \in \langle A\sqcap\rangle^*\mathscr{S}\mathcal{A}$. □

Proposition 707. $\mathcal{X} \div \operatorname{Base}(\mathcal{X}) = \mathcal{X}$.

Proof. Because $X \sqcap \operatorname{Base}(\mathcal{X}) = X$ for $X \in \mathcal{X}$. □

Proposition 708. $(\mathcal{X} \div A) \div B = \mathcal{X} \div B$ if $B \sqsubseteq A$.

Proof.

$$(\mathcal{X} \div A) \div B = \bigsqcap\left\{\frac{\uparrow^B (Y \sqcap B)}{Y \in \bigsqcap\left\{\frac{\uparrow^A(X\sqcap A)}{X\in\mathcal{X}}\right\}}\right\} = \bigsqcap\left\{\frac{\uparrow^B (X \sqcap A)}{X \in \mathcal{X}}\right\}\sqcap \uparrow^B B =$$

$$\bigsqcap\left\{\frac{\uparrow^B (X \sqcap A \sqcap B)}{X \in \mathcal{X}}\right\} = \bigsqcap\left\{\frac{\uparrow^B (X \sqcap B)}{X \in \mathcal{X}}\right\} = \mathcal{X} \div B.$$

□

Proposition 709. If $A \in \mathcal{A}$ then $\mathcal{A} \div A = \mathcal{A} \cap \mathscr{P}A$.

Proof. $\mathcal{A} \div A = \langle A\sqcap\rangle^*\mathscr{S}\mathcal{A} = \langle A\sqcap\rangle^*\left\{\frac{K\in\mathfrak{Z}}{\exists X\in\mathcal{A}:X\sqsubseteq K}\right\} = \left\{\frac{K\in\mathfrak{Z}}{K\in\mathcal{A}\wedge K\in\mathscr{P}A}\right\} = \mathcal{A} \cap \mathscr{P}A$. □

Proposition 710. Let filters $\mathcal{X}$ and $\mathcal{Y}$ be such that $\operatorname{Base}(\mathcal{X}) = \operatorname{Base}(\mathcal{Y}) = B$. Then $\mathcal{X} \div C = \mathcal{Y} \div C \Leftrightarrow \mathcal{X} = \mathcal{Y}$ for every $\mathfrak{Z} \ni C \sqsupseteq B$.

Proof. $\mathcal{X} \div C = \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \cup \left\{\frac{K\in\mathscr{P}C}{K\sqsupseteq B}\right\} = \mathcal{Y} \cup \left\{\frac{K\in\mathscr{P}C}{K\sqsupseteq B}\right\} \Leftrightarrow \mathcal{X} = \mathcal{Y}$. □

5.39.2. Equivalence of filters.

Definition 711. Two filters $\mathcal{A}$ and $\mathcal{B}$ (with possibly different base sets) are equivalent ($\mathcal{A} \sim \mathcal{B}$) iff there exists an $X \in \mathfrak{Z}$ such that $X \in \mathcal{A}$ and $X \in \mathcal{B}$ and $\mathscr{P}X \cap \mathcal{A} = \mathscr{P}X \cap \mathcal{B}$.

Proposition 712. $\mathcal{X}$ and $\mathcal{Y}$ are equivalent iff ($\mathcal{X} \sim \mathcal{Y}$) iff $\mathcal{Y} = \mathcal{X} \div \operatorname{Base}(\mathcal{Y})$ and $\mathcal{X} = \mathcal{Y} \div \operatorname{Base}(\mathcal{X})$.

Proof.

$\Rightarrow$. Suppose $\mathcal{X} \sim \mathcal{Y}$ that is there exists a set P such that $\mathscr{P}P \cap \mathcal{X} = \mathscr{P}P \cap \mathcal{Y}$ and $P \in \mathcal{X}$, $P \in \mathcal{Y}$. Then $\mathcal{X} \div \operatorname{Base}(\mathcal{Y}) = (\mathscr{P}P \cap \mathcal{X}) \cup \left\{\frac{K\in\mathscr{P}\operatorname{Base}(\mathcal{Y})}{K\sqsupseteq P}\right\} = (\mathscr{P}P \cap \mathcal{Y}) \cup \left\{\frac{K\in\mathscr{P}\operatorname{Base}(\mathcal{Y})}{K\sqsupseteq P}\right\} = \mathcal{Y}$. So $\mathcal{X} \div \operatorname{Base}(\mathcal{Y}) = \mathcal{Y}$, $\mathcal{Y} \div \operatorname{Base}(\mathcal{X}) = \mathcal{X}$ is similar.

$\Leftarrow$. We have $\mathcal{Y} = (\mathcal{Y} \div \operatorname{Base}(\mathcal{X})) \div \operatorname{Base}(\mathcal{Y})$.

Thus as easy to show $\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y}) \in \mathcal{Y}$ and similarly $\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y}) \in \mathcal{X}$.

It's enough to show $\mathcal{X} \div (\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y})) = \mathcal{Y} \div (\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y}))$ because for every $P \in \mathcal{X}, \mathcal{Y}$ we have $\mathcal{X} \cap \mathscr{P}P = \mathcal{X} \div P = (\mathcal{X} \div (\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y}))) \div P$ and similarly $\mathcal{Y} \cap \mathscr{P}P = (\mathcal{Y} \div (\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y}))) \div P$. But it follows from the conditions and proposition 708.

□

Proposition 713. If two filters with the same base are equivalent they are equal.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two filters and $\mathscr{P}X \cap \mathcal{A} = \mathscr{P}X \cap \mathcal{B}$ for some set X such that $X \in \mathcal{A}$ and $X \in \mathcal{B}$, and $\operatorname{Base}(\mathcal{A}) = \operatorname{Base}(\mathcal{B})$. Then

$$\mathcal{A} = (\mathscr{P}X \cap \mathcal{A}) \cup \left\{ \frac{Y \in D \operatorname{Base}(\mathcal{A})}{Y \sqsupseteq X} \right\} = (\mathscr{P}X \cap \mathcal{B}) \cup \left\{ \frac{Y \in D \operatorname{Base}(\mathcal{B})}{Y \sqsupseteq X} \right\} = \mathcal{B}.$$

□

Proposition 714. If $A \in \mathscr{S}\mathcal{A}$ then $\mathcal{A} \div A \sim \mathcal{A}$.

Proof.

$$\begin{aligned} (\mathcal{A} \div A) \cap \mathscr{P}(A \sqcap \operatorname{Base}(\mathcal{A})) &= \\ \mathscr{S}\mathcal{A} \cap \mathscr{P}A \cap \mathscr{P}(A \sqcap \operatorname{Base}(\mathcal{A})) &= \\ \mathscr{S}\mathcal{A} \cap \mathscr{P}(A \sqcap \operatorname{Base}(\mathcal{A})) &= \mathcal{A} \cap \mathscr{P}(A \sqcap \operatorname{Base}(\mathcal{A})). \end{aligned}$$

Thus $\mathcal{A} \div A \sim \mathcal{A}$ because $A \sqcap \operatorname{Base}(\mathcal{A}) \sqsupseteq X \in \mathcal{A}$ for some $X \in \mathcal{A}$ and

$$A \sqcap \operatorname{Base}(\mathcal{A}) \sqsupseteq X \sqcap \operatorname{Base}(\mathcal{A}) \in \mathcal{A} \div A.$$

□

Proposition 715. $\sim$ is an equivalence relation.

Proof.

Reflexivity. Obvious.

Symmetry. Obvious.

Transitivity. Let $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$ for some filters $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. Then there exist a set X such that $X \in \mathcal{A}$ and $X \in \mathcal{B}$ and $\mathscr{P}X \cap \mathcal{A} = \mathscr{P}X \cap \mathcal{B}$ and a set Y such that $Y \in \mathcal{B}$ and $Y \in \mathcal{C}$ and $\mathscr{P}Y \cap \mathcal{B} = \mathscr{P}Y \cap \mathcal{C}$. So $X \sqcap Y \in \mathcal{A}$ because

$$\mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{A} = \mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{B} = \mathscr{P}(X \sqcap Y) \cap \mathcal{B} \supseteq \{X \sqcap Y\} \cap \mathcal{B} \ni X \sqcap Y.$$

Similarly we have $X \sqcap Y \in \mathcal{C}$. Finally

$$\mathscr{P}(X \sqcap Y) \cap \mathcal{A} = \mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{A} = \mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{B} = \mathscr{P}X \cap \mathscr{P}Y \cap \mathcal{B} = \mathscr{P}X \cap \mathscr{P}Y \cap \mathcal{C} = \mathscr{P}(X \sqcap Y) \cap \mathcal{C}.$$

□

Definition 716. I will call equivalence classes as *unfixed filters*.

Remark 717. The word "unfixed" is meant to negate "fixed" (having a particular base) filters.

PROPOSITION 718. $\mathcal{A} \sim \mathcal{B}$ iff $\mathscr{S}\mathcal{A} = \mathscr{S}\mathcal{B}$ for every filters $\mathcal{A}$, $\mathcal{B}$ on sets.[3]

PROOF. Let $\mathcal{A} \sim \mathcal{B}$. Then there is a set P such that $P \in \mathcal{A}$, $P \in \mathcal{B}$ and $\mathcal{A} \cap \mathscr{P}P = \mathcal{B} \cap \mathscr{P}P$. So $\mathscr{S}\mathcal{A} = (\mathcal{A} \cap \mathscr{P}P) \cup \left\{ \frac{K \in \mathfrak{Z}}{K \sqsupseteq P} \right\}$. Similarly $\mathscr{S}\mathcal{B} = (\mathcal{B} \cap \mathscr{P}P) \cup \left\{ \frac{K \in \mathfrak{Z}}{K \sqsupseteq P} \right\}$. Combining, we have $\mathscr{S}\mathcal{A} = \mathscr{S}\mathcal{B}$.

Let now $\mathscr{S}\mathcal{A} = \mathscr{S}\mathcal{B}$. Take $K \in \mathscr{S}\mathcal{A} = \mathscr{S}\mathcal{B}$. Then $\mathcal{A} \div K = \mathcal{B} \div K$ and thus (proposition 714) $\mathcal{A} \sim \mathcal{A} \div K = \mathcal{B} \div K \sim \mathcal{B}$, so having $\mathcal{A} \sim \mathcal{B}$. □

PROPOSITION 719. $\mathcal{A} \sim \mathcal{B} \Rightarrow \mathcal{A} \div B = \mathcal{B} \div B$ for every filters $\mathcal{A}$ and $\mathcal{B}$ and set B.

PROOF. $\mathcal{A} \div B = \langle B \sqcap \rangle^* \mathscr{S}\mathcal{A} = \langle B \sqcap \rangle^* \mathscr{S}\mathcal{B} = \mathcal{B} \div B$. □

5.39.3. Poset of unfixed filters.

LEMMA 720. Let filters $\mathcal{X}$ and $\mathcal{Y}$ be such that $\operatorname{Base}(\mathcal{X}) = \operatorname{Base}(\mathcal{Y}) = B$. Then $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}$ for every set $C \supseteq B$.

PROOF.

$$\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \div C \supseteq \mathcal{Y} \div C \Leftrightarrow$$
$$\mathcal{X} \cup \left\{ \frac{K \in \mathscr{P}C}{K \sqsupseteq B} \right\} \supseteq \mathcal{Y} \cup \left\{ \frac{K \in \mathscr{P}C}{K \sqsupseteq B} \right\} \Leftrightarrow \mathcal{X} \supseteq \mathcal{Y} \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}.$$

□

PROPOSITION 721. $\mathcal{X} \sqsubseteq \mathcal{Y} \Rightarrow \mathcal{X} \div B \sqsubseteq \mathcal{Y} \div B$ for every filters $\mathcal{X}$, $\mathcal{Y}$ with the same base and set B.

PROOF. $\mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \mathcal{X} \supseteq \mathcal{Y} \Rightarrow \mathcal{X} \div B \supseteq \mathcal{Y} \div B \Leftrightarrow \mathcal{X} \div B \sqsubseteq \mathcal{Y} \div B$. □

Define order of unfixed filters using already defined order of filters of a fixed base:

DEFINITION 722. $\mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \exists x \in \mathcal{X}, y \in \mathcal{Y} : (\operatorname{Base}(x) = \operatorname{Base}(y) \wedge x \sqsubseteq y)$ for unfixed filters $\mathcal{X}$, $\mathcal{Y}$.

Proposition 718 allows to define:

DEFINITION 723. $\mathscr{S}\mathcal{A} = \mathscr{S}a$ for every $a \in \mathcal{A}$ for every unfixed filter $\mathcal{A}$.

THEOREM 724. $\mathscr{S}$ is an order-isomorphism from the poset of unfixed filters to the poset of filters on $\mathfrak{Z}$.

PROOF. We already know that $\mathscr{S}$ is an order embedding. It remains to prove that it is a surjection.

Let $\mathcal{Y}$ be a filter on $\mathfrak{Z}$. Take $\mathfrak{Z} \ni X \in \mathcal{Y}$. Then $\langle X \sqcap \rangle^* \mathcal{Y}$ is a filter on X and $\mathscr{S}[\langle X \sqcap \rangle^* \mathcal{Y}] = \mathscr{S}\langle X \sqcap \rangle^* \mathcal{Y} = \mathcal{Y}$. We have proved that it is a surjection. □

LEMMA 725. $\mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \mathscr{S}\mathcal{X} \sqsubseteq \mathscr{S}\mathcal{Y}$ for every unfixed filters $\mathcal{X}$, $\mathcal{Y}$.

PROOF.

$\Rightarrow$. Suppose $\mathcal{X} \sqsubseteq \mathcal{Y}$. Then there exist $x \in \mathcal{X}$, $y \in \mathcal{Y}$ such that $\operatorname{Base}(x) = \operatorname{Base}(y)$ and $x \sqsubseteq y$. Then $\mathscr{S}\mathcal{X} = \mathscr{S}x \sqsubseteq \mathscr{S}y = \mathscr{S}\mathcal{Y}$.

$\Leftarrow$. Suppose $\mathscr{S}\mathcal{X} \sqsubseteq \mathscr{S}\mathcal{Y}$. Then there are $x \in \mathcal{X}$, $y \in \mathcal{Y}$ such that $\mathscr{S}x \sqsubseteq \mathscr{S}y$. Consequently $\mathscr{S}x' \sqsubseteq \mathscr{S}y'$ for $x' = x \div (\operatorname{Base}(x) \sqcup \operatorname{Base}(y))$, $y' = y \div (\operatorname{Base}(x) \sqcup \operatorname{Base}(y))$. So we have $x' \in \mathcal{X}$, $y' \in \mathcal{Y}$, $\operatorname{Base}(x') = \operatorname{Base}(y')$ and $x' \sqsubseteq y'$, thus $\mathcal{X} \sqsubseteq \mathcal{Y}$.

[3] Use this proposition to shorten proofs of other theorem about equivalence of filters? (Our proof uses transitivity of equivalence of filters. So we can't use it to prove that it is an equivalence relation, to avoid circular proof.)

□

THEOREM 726. $\sqsubseteq$ on the set of unfixed filters is a poset.

PROOF.

Reflexivity. From the previous theorem.
Transitivity. From the previous theorem.
Antisymmetry. Suppose $\mathcal{X} \sqsubseteq \mathcal{Y}$ and $\mathcal{Y} \sqsubseteq \mathcal{X}$. Then $\mathscr{S}\mathcal{X} \sqsubseteq \mathscr{S}\mathcal{Y}$ and $\mathscr{S}\mathcal{Y} \sqsubseteq \mathscr{S}\mathcal{X}$. Thus $\mathscr{S}\mathcal{X} = \mathscr{S}\mathcal{Y}$ and so $\mathscr{S}x = \mathscr{S}y$ for some $x \in \mathcal{X}$, $y \in \mathcal{Y}$. Consequently $\mathscr{S}(x \div B) = \mathscr{S}(y \div B)$ for $B = \operatorname{Base}(x) \sqcup \operatorname{Base}(y)$. Thus $x \div B = y \div B$ and so $x \sim y$, thus $\mathcal{X} = \mathcal{Y}$.

□

THEOREM 727. $[x] \sqsubseteq [y] \Leftrightarrow x \sqsubseteq y$ for filters x and y with the same base set.

PROOF.

$\Leftarrow$. Obvious.
$\Rightarrow$. Let $\operatorname{Base}(x) = \operatorname{Base}(y) = B$. Suppose $[x] \sqsubseteq [y]$. Then there exist $x' \sim x$ and $y' \sim y$ such that $C = \operatorname{Base}(x') = \operatorname{Base}(y')$ (for some set C) and $x' \sqsubseteq y'$.
We have by the lemma $x' \div (B \sqcup C) \sqsubseteq y' \div (B \sqcup C)$.
But $x' \div (B \sqcup C) = x \div (B \sqcup C)$ and $y' \div (B \sqcup C) = y \div (B \sqcup C)$. So $x \div (B \sqcup C) \sqsubseteq y \div (B \sqcup C)$ and thus again applying the lemma $x \sqsubseteq y$.

□

5.39.4. Rebase of unfixed filters. Proposition 719 allows to define:

DEFINITION 728. $\mathcal{A} \div B = a \div B$ for an unfixed filter $\mathcal{A}$ and arbitrary $a \in \mathcal{A}$.

PROPOSITION 729. $\mathcal{X} \sqsubseteq \mathcal{Y} \Rightarrow \mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C$ for every unfixed filters $\mathcal{X}$, $\mathcal{Y}$ and set C.

PROOF. Let $\mathcal{X} \sqsubseteq \mathcal{Y}$. Then there are $x \in \mathcal{X}$, $y \in \mathcal{Y}$ such that $\operatorname{Base}(x) = \operatorname{Base}(y)$ and $x \sqsubseteq y$. Then by proved above $x \div C \sqsubseteq y \div C$ what is equivalent to $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C$. □

PROPOSITION 730. If $C \in \mathscr{S}\mathcal{X}$ and $C \in \mathscr{S}\mathcal{Y}$ for unfixed filters $\mathcal{X}$ and $\mathcal{Y}$ then $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}$.

PROOF.

$\Leftarrow$. Previous proposition.
$\Rightarrow$. Let $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C$. We have some $x \in \mathcal{X}$, $y \in \mathcal{Y}$, such that $\operatorname{Base}(x) = \operatorname{Base}(y)$ and $x \div C \sqsubseteq y \div C$. So $\mathscr{S}(x \div C) \sqsubseteq \mathscr{S}(y \div C)$. But $\mathscr{S}(x \div C) \sim x$ and $\mathscr{S}(y \div C) \sim y$. Thus $\mathscr{S}x \sqsubseteq \mathscr{S}y$ that is $x \sqsubseteq y$ and so $\mathcal{X} \sqsubseteq \mathcal{Y}$.

□

OBVIOUS 731. $(\mathcal{X} \div A) \div B = \mathcal{X} \div B$ if $B \sqsubseteq A$ for every unfixed filter $\mathcal{X}$ and sets A, B.

OBVIOUS 732. $\mathcal{A} \div B = \langle B \sqcap \rangle^* \mathscr{S}\mathcal{A}$ for every unfixed filter $\mathcal{A}$.

OBVIOUS 733. If $A \in \mathscr{S}\mathcal{A}$ then $\mathcal{A} \div A \in \mathcal{A}$ for every unfixed filter $\mathcal{A}$.

PROPOSITION 734. If $C \in \mathscr{S}\mathcal{X}$ and $C \in \mathscr{S}\mathcal{Y}$ for unfixed filters $\mathcal{X}$ and $\mathcal{Y}$ then $\mathcal{X} \div C = \mathcal{Y} \div C \Leftrightarrow \mathcal{X} = \mathcal{Y}$.

PROOF. The backward implication is obvious. Let now $\mathcal{X} \div C = \mathcal{Y} \div C$. Take $x \in \mathcal{X}$, $y \in \mathcal{Y}$. We have $\mathcal{X} \div C = x \div C = (x \div B) \div C$ for $B = C \sqcup \operatorname{Base}(x) \sqcup \operatorname{Base}(y)$. Similary $\mathcal{Y} \div C = (y \div B) \div C$. Thus $(x \div B) \div C = (y \div B) \div C$ and thus $x \div B = y \div B$, so $x \sim y$ that is $\mathcal{X} = \mathcal{Y}$. □

PROPOSITION 735. $\mathcal{A} \div A = \bigsqcap\left\{\frac{\uparrow^A (X \sqcap A)}{X \in \mathscr{S}\mathcal{A}}\right\}$ for every unfixed filter $\mathcal{A}$.

PROOF. Take $a \in \mathcal{A}$.

$$\bigsqcap\left\{\frac{\uparrow^A (X \sqcap A)}{X \in \mathscr{S}\mathcal{A}}\right\} = \bigsqcap\left\{\frac{\uparrow^A (X \sqcap A \sqcap \operatorname{Base}(a))}{X \in \mathscr{S}\mathcal{A}}\right\} =$$
$$\bigsqcap\left\{\frac{\uparrow^A (X \sqcap A)}{X \in \mathscr{S}\mathcal{A} \cap \mathscr{P}\operatorname{Base}(a)}\right\} = \bigsqcap\left\{\frac{\uparrow^A (X \sqcap A)}{X \in \mathscr{S}a \cap \mathscr{P}\operatorname{Base}(a)}\right\} =$$
$$\bigsqcap\left\{\frac{\uparrow^A (X \sqcap A)}{X \in a}\right\} = a \div A = \mathcal{A} \div A.$$

□

5.39.5. The diagram for unfixed filters. Fix a set B.

LEMMA 736. $\mathcal{X} \mapsto \mathcal{X} \div B$ and $x \mapsto [x]$ are mutually inverse order isomorphisms between $\left\{\frac{\text{unfixed filter } \mathcal{X}}{B \in \mathscr{S}\mathcal{X}}\right\}$ and $\mathfrak{F}(DB)$.

PROOF. First, $\mathcal{X} \div B \in \mathfrak{F}(DB)$ for $\mathcal{X} \in \left\{\frac{\text{unfixed filter } \mathcal{X}}{B \in \mathscr{S}\mathcal{X}}\right\}$ and $[x] \in \left\{\frac{\text{unfixed filter } \mathcal{X}}{B \in \mathscr{S}\mathcal{X}}\right\}$ for $x \in \mathfrak{F}(DB)$.

Suppose $\mathcal{X}_0 \in \left\{\frac{\text{unfixed filter } \mathcal{X}}{B \in \mathscr{S}\mathcal{X}}\right\}$, $x = \mathcal{X}_0 \div B$, and $\mathcal{X}_1 = [x]$. We will prove $\mathcal{X}_0 = \mathcal{X}_1$. Really, $x \in \mathcal{X}_1$, $x = k \div B$ for $k \in \mathcal{X}_0$, $x \sim k$, thus $x \in \mathcal{X}_0$. So $\mathcal{X}_0 = \mathcal{X}_1$.

Suppose $x_0 \in \mathfrak{F}(DB)$, $\mathcal{X} = [x_0]$, $x_1 = \mathcal{X} \div B$. We will prove $x_0 = x_1$. Really, $x_1 = x_0 \div B$. So $x_1 = x_0$ because $\operatorname{Base}(x_0) = \operatorname{Base}(x_1) = B$.

So we proved that they are mutually inverse bijections. That they are order preserving is obvious. □

LEMMA 737. $\mathscr{S}$ and $\mathcal{X} \mapsto \langle B\sqcap\rangle^*\mathcal{X} = \mathcal{X} \cap \mathscr{P}B$ are mutually inverse order isomorphisms between $\mathfrak{F}(DB)$ and $\left\{\frac{\mathcal{X} \in \mathfrak{F}(\mathfrak{Z})}{B \in \mathcal{X}}\right\}$.

PROOF. First, $\mathscr{S}x \in \left\{\frac{\mathcal{X} \in \mathfrak{F}(\mathfrak{Z})}{B \in \mathcal{X}}\right\}$ for $x \in \mathfrak{F}(DB)$ because of theorem 724 and $\langle B\sqcap\rangle^*\mathcal{X} \in \mathfrak{F}(DB)$ obviously.

Let's prove $\langle B\sqcap\rangle^*\mathcal{X} = \mathcal{X} \cap \mathscr{P}B$. If $X \in \langle B\sqcap\rangle^*\mathcal{X}$ then $X \in \mathcal{X}$ (because $B \in \mathcal{X}$) and $X \in \mathscr{P}B$. So $X \in \mathcal{X} \cap \mathscr{P}B$. If $X \in \mathcal{X} \cap \mathscr{P}B$ then $X = B \cap X \in \langle B\sqcap\rangle^*\mathcal{X}$.

Let $x_0 \in \mathfrak{F}(DB)$, $\mathcal{X} = \mathscr{S}x$, and $x_1 = \langle B\sqcap\rangle^*\mathcal{X}$. Then obviously $x_0 = x_1$.

Let now $\mathcal{X}_0 \in \left\{\frac{\mathcal{X} \in \mathfrak{F}(\mathfrak{Z})}{B \in \mathcal{X}}\right\}$, $x = \langle B\sqcap\rangle^*\mathcal{X}_0$, and $\mathcal{X}_1 = \mathscr{S}x$. Then $\mathcal{X}_1 = \mathcal{X}_0 \cup \left\{\frac{K \in \mathfrak{Z}}{K \sqsupseteq B}\right\} = \mathcal{X}_0$.

So we proved that they are mutually inverse bijections. That they are order preserving is obvious. □

THEOREM 738. The diagram at the figure 7 (with the horizontal "unnamed" arrow *defined* as the inverse isomorphism of its opposite arrow) is a commutative diagram (in category **Set**), every arrow in this diagram is an isomorphism. Every cycle in this diagram is an identity (therefore "parallel" arrows are mutually inverse). The arrows preserve order.

PROOF. It's proved above, that all morphisms (except the "unnamed" arrow, which is the inverse morphism by definition) depicted on the diagram are bijections and the depicted "opposite" morphisms are mutually inverse.

That arrows preserve order is obvious.

It remains to apply lemma 196 (taking into account the proof of theorem 724). □

$$\begin{tikzcd}
 & \mathfrak{F}(DB) \arrow[dl, shift left, "\mathscr{S}"] \arrow[dr, shift left, "\mathcal{X}\mapsto\mathcal{X}\div B"] & \\
\left\{\frac{\mathcal{X}\in\mathfrak{F}(\mathfrak{Z})}{B\in\mathcal{X}}\right\} \arrow[ur, shift left, "\mathcal{X}\mapsto\langle B\sqcap\rangle^*\mathcal{X}=\mathcal{X}\cap\mathscr{P}B"] \arrow[rr, shift left] & & \left\{\frac{\text{unfixed filter } \mathcal{X}}{B\in\mathscr{S}\mathcal{X}}\right\} \arrow[ul, shift left, "x\mapsto[x]"] \arrow[ll, shift left, "\mathscr{S}"]
\end{tikzcd}$$

FIGURE 7

5.39.6. The lattice of unfixed filters.

THEOREM 739. Every nonempty set of unfixed filters has an infimum, provided that the lattice $\mathfrak{Z}$ is distributive.

PROOF. Theorem 520. □

THEOREM 740. Every bounded above set of unfixed filters has a supremum.

PROOF. Theorem 515 for nonempty sets of unfixed filters. The join $\bigsqcup\emptyset = [\bot]$ for the least filter $\bot \in \mathfrak{Z}(DA)$ for arbitrary $A \in \mathfrak{Z}$. □

COROLLARY 741. If $\mathfrak{Z}$ is the set of small sets, then every small set of unfixed filters has a supremum.

PROOF. Let S be a set of filters on $\mathfrak{Z}$. Then $T_{\mathcal{X}} \in \mathcal{X}$ is a small set for every $\mathcal{X} \in S$. Thus $\left\{\frac{T_{\mathcal{X}}}{\mathcal{X}\in S}\right\}$ is small set and thus $T = \bigcup\left\{\frac{T_{\mathcal{X}}}{\mathcal{X}\in S}\right\}$ is small set. Take the filter $\mathcal{T} = \uparrow T$. Then $\mathcal{T}$ is an upper bound of S and we can apply the theorem. □

OBVIOUS 742. The poset of unfixed filters for the lattice of small sets is bounded below (but not above).

PROPOSITION 743. The set of unfixed filters forms a co-brouwerian (and thus distributive) lattice, provided that $\mathfrak{Z}$ is distributive lattice which is an ideal base.

PROOF. Corollary 531. □

5.39.7. Principal unfixed filters and filtrator of unfixed filters.

DEFINITION 744. *Principal* unfixed filter is an unfixed filter corresponding to a principal filter on the poset $\mathfrak{Z}$.

DEFINITION 745. The *filtrator of unfixed filters* is the filtrator whose base are unfixed filters and whose core are principal unfixed filters.

We will equate principal unfixed filters with corresponding sets.

THEOREM 746. If we add principal filters on DB, principal filters on $\mathfrak{Z}$ containing B, and above defined principal unfixed filters corresponding to them to appropriate nodes of the diagram 7, then the diagram turns into a commutative diagram of isomorphisms between filtrators. (I will not draw the modified diagram for brevity.)

Every arrow of this diagram is an isomorphism between filtrators, every cycle in the diagram is identity.

PROOF. We need to prove only that principal filters on B and principal filters on $\mathfrak{Z}$ containing B correspond to each other by the isomorphisms of the diagram. But that's obvious. □

OBVIOUS 747. The filtrator of unfixed filters is a primary filtrator.

OBVIOUS 748. The filtrator of unfixed filters is down-aligned.

PROPOSITION 749. The filtrator of unfixed filters is

1°. filtered;
2°. with join-closed core.

PROOF. Theorem 534. □

PROPOSITION 750. The filtrator of unfixed filters is with binarily meet-closed core.

PROOF. Corollary 536. □

PROPOSITION 751. The filtrator of unfixed filters is with separable core.

PROOF. Theorem 537. □

PROPOSITION 752. $\operatorname{Cor}\mathcal{X}$ and $\operatorname{Cor}'\mathcal{X}$ are defined for every unfixed filter $\mathcal{X}$ and $\operatorname{Cor}\mathcal{X} = \operatorname{Cor}'\mathcal{X}$, provided that every DA is a complete lattice.

PROOF. $\operatorname{Cor}\mathcal{X}$ and $\operatorname{Cor}'\mathcal{X}$ exists because of the above isomorphism.
$\operatorname{Cor}'\mathcal{X} = \operatorname{Cor}\mathcal{X}$ by theorem 545. □

OBVIOUS 753. $\operatorname{Cor}\mathcal{X} = \operatorname{Cor}'\mathcal{X} = \bigcap\mathcal{X}$ for every filter $\mathcal{X} \in \mathfrak{F}(\text{small sets})$.

PROPOSITION 754. $\operatorname{atoms}\bigsqcap S = \bigcap\langle\operatorname{atoms}\rangle^* S$ whenever $\bigsqcap S$ is defined.

PROOF. Theorem 108. □

PROPOSITION 755. $\operatorname{atoms}(\mathcal{A}\sqcup\mathcal{B}) = \operatorname{atoms}\mathcal{A}\cup\operatorname{atoms}\mathcal{B}$ for unfixed filters $\mathcal{A}$, $\mathcal{B}$, whenever $\mathfrak{Z}$ is a distributive lattice which is an ideal base.

PROOF. Proposition 557. □

PROPOSITION 756. $\partial\mathcal{X}$ is a free star for every unfixed filter $\mathcal{X}$, whenever $\mathfrak{Z}$ is a distributive lattice which is an ideal base which has a least element.

PROOF. Theorem 566. □

PROPOSITION 757. The poset of unfixed filters is an atomistic lattice if every DA (for $A \in \mathfrak{A}$) is an atomistic lattice.

PROOF. Easily follows from 738 by isomorphism. □

PROPOSITION 758. The poset of unfixed filters is a strongly separable lattice if every DA (for $A \in \mathfrak{A}$) is an atomistic lattice.

PROOF. Theorem 234. □

PROPOSITION 759. $\operatorname{Cor}\mathcal{X} = \bigsqcup(\mathfrak{Z}\cap\operatorname{atoms}^{\text{unfixed filters}})$ for every unfixed filter $\mathcal{X}$ if every DA (for $A \in \mathfrak{A}$) is an atomistic lattice.

PROOF. Theorem 599. □

PROPOSITION 760. $\operatorname{Cor}(\mathcal{A}\sqcap\mathcal{B}) = \operatorname{Cor}\mathcal{A}\sqcap\operatorname{Cor}\mathcal{B}$ for every unfixed filters $\mathcal{A}$, $\mathcal{B}$, provided every DA (for $A \in \mathfrak{A}$) is a complete lattice.

PROOF. Theorem 601. □

PROPOSITION 761. $\operatorname{Cor}\bigsqcap^{\mathfrak{A}} S = \bigsqcap^{\mathfrak{Z}}\langle\operatorname{Cor}\rangle^* S$ for the filtrator of unfixed filters for every nonempty set S of unfixed filters, provided every DA (for $A \in \mathfrak{A}$) is a complete lattice.

PROOF. Theorem 602. □

PROPOSITION 762. $\operatorname{Cor}(\mathcal{A} \sqcup^{\mathfrak{A}} \mathcal{B}) = \operatorname{Cor} \mathcal{A} \sqcup^{\mathfrak{Z}} \operatorname{Cor} \mathcal{B}$ for the filtrator of unfixed filters for every unfixed filters $\mathcal{A}$, and $\mathcal{B}$, provided every DA (for $A \in \mathfrak{A}$) is a complete atomistic distributive lattice.

PROOF. Can be easily deduced from theorem 603 and the triangular diagram (above) of isomorphic filtrators. □

CONJECTURE 763. The theorem 614 holds for unfixed filters, too.

It is expected to be easily provable using isomorphisms from the triangular diagram.

CHAPTER 6

Common knowledge, part 2 (topology)

In this chapter I describe basics of the theory known as *general topology*. Starting with the next chapter after this one I will describe generalizations of customary objects of general topology described in this chapter.

The reason why I've written this chapter is to show to the reader kinds of objects which I generalize below in this book. For example, funcoids and a generalization of proximity spaces, and funcoids are a generalization of pretopologies. To understand the intuitive meaning of funcoids one needs first know what are proximities and what are pretopologies.

Having said that, customary topology is *not* used in my definitions and proofs below. It is just to feed your intuition.

6.1. Metric spaces

The theory of topological spaces started immediately with the definition would be completely non-intuitive for the reader. It is the reason why I first describe metric spaces and show that metric spaces give rise for a topology (see below). Topological spaces are understandable as a generalization of topologies induced by metric spaces.

Metric spaces is a formal way to express the notion of *distance*. For example, there are distance $|x - y|$ between real numbers x and y, distance between points of a plane, etc.

Definition 764. A *metric space* is a set U together with a function $d : U \times U \to \mathbb{R}$ (*distance* or *metric*) such that for every $x, y, z \in U$:

1°. $d(x, y) \geq 0$;
2°. $d(x, y) = 0 \Leftrightarrow x = y$;
3°. $d(x, y) = d(y, x)$ (*symmetry*);
4°. $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality*).

Exercise 765. Show that the Euclid space $\mathbb{R}^n$ (with the standard distance) is a metric space for every $n \in \mathbb{N}$.

Definition 766. *Open ball* of *radius* $r > 0$ centered at point $a \in U$ is the set

$$B_r(a) = \left\{ \frac{x \in U}{d(a, x) < r} \right\}.$$

Definition 767. *Closed ball* of *radius* $r > 0$ centered at point $a \in U$ is the set

$$B_r[a] = \left\{ \frac{x \in U}{d(a, x) \leq r} \right\}.$$

One example of use of metric spaces: *Limit* of a sequence x in a metric space can be defined as a point y in this space such that

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n > N : d(x_n, y) < \epsilon.$$

6.1.1. Open and closed sets.

DEFINITION 768. A set A in a metric space is called *open* when $\forall a \in A \exists r > 0 : B_r(a) \subseteq A$.

DEFINITION 769. A set A in a metric space is closed when its complement $U \setminus A$ is open.

EXERCISE 770. Show that: closed intervals on real line are closed sets, open intervals are open sets.

EXERCISE 771. Show that open balls are open and closed balls are closed.

DEFINITION 772. Closure $\operatorname{cl}(A)$ of a set A in a metric space is the set of points y such that

$$\forall \epsilon > 0 \exists a \in A : d(y, a) < \epsilon.$$

PROPOSITION 773. $\operatorname{cl}(A) \supseteq A$.

PROOF. It follows from $d(a, a) = 0 < \epsilon$. □

EXERCISE 774. Prove $\operatorname{cl}(A \cup B) = \operatorname{cl}(A) \cup \operatorname{cl}(B)$ for every subsets A and B of a metric space.

6.2. Pretopological spaces

Pretopological space can be defined in two equivalent ways: a *neighborhood system* or a *preclosure operator*. To be more clear I will call *pretopological space* only the first (neighborhood system) and the second call a *preclosure space*.

DEFINITION 775. *Pretopological space* is a set U together with a filter $\Delta(x)$ on U for every $x \in U$, such that $\uparrow^U \{x\} \sqsubseteq \Delta(x)$. Δ is called a *pretopology* on U. Elements of $\operatorname{up} \Delta(x)$ are called *neighborhoods* of point x.

DEFINITION 776. *Preclosure* on a set U is a unary operation cl on $\mathscr{P}U$ such that for every $A, B \in \mathscr{P}U$:

1°. $\operatorname{cl}(\emptyset) = \emptyset$;
2°. $\operatorname{cl}(A) \supseteq A$;
3°. $\operatorname{cl}(A \cup B) = \operatorname{cl}(A) \cup \operatorname{cl}(B)$.

I call a preclosure together with a set U as *preclosure space*.

THEOREM 777. Small pretopological spaces and small preclosure spaces bijectively correspond to each other by the formulas:

$$\operatorname{cl}(A) = \left\{ \frac{x \in U}{A \in \partial\Delta(x)} \right\}; \tag{3}$$

$$\operatorname{up} \Delta(x) = \left\{ \frac{A \in \mathscr{P}U}{x \notin \operatorname{cl}(U \setminus A)} \right\}. \tag{4}$$

PROOF. First let's prove that cl defined by formula (3) is really a preclosure.

$\operatorname{cl}(\emptyset) = \emptyset$ is obvious. If $x \in A$ then $A \in \partial\Delta(x)$ and so $\operatorname{cl}(A) \supseteq A$. $\operatorname{cl}(A \cup B) = \left\{ \frac{x \in U}{A \cup B \in \partial\Delta(x)} \right\} = \left\{ \frac{x \in U}{A \in \partial\Delta(x) \vee B \in \partial\Delta(x)} \right\} = \operatorname{cl}(A) \cup \operatorname{cl}(B)$. So, it is really a preclosure.

Next let's prove that Δ defined by formula (4) is a pretopology. That $\operatorname{up} \Delta(x)$ is an upper set is obvious. Let $A, B \in \operatorname{up} \Delta(x)$. Then $x \notin \operatorname{cl}(U \setminus A) \wedge x \notin \operatorname{cl}(U \setminus B)$; $x \notin \operatorname{cl}(U \setminus A) \cup \operatorname{cl}(U \setminus B) = \operatorname{cl}((U \setminus A) \cup (U \setminus B)) = \operatorname{cl}(U \setminus (A \cap B))$; $A \cap B \in \operatorname{up} \Delta(x)$. We have proved that $\Delta(x)$ is a filter object.

Let's prove $\uparrow^U \{x\} \sqsubseteq \Delta(x)$. If $A \in \operatorname{up} \Delta(x)$ then $x \notin \operatorname{cl}(U \setminus A)$ and consequently $x \notin U \setminus A$; $x \in A$; $A \in \operatorname{up} \uparrow^U \{x\}$. So $\uparrow^U \{x\} \sqsubseteq \Delta(x)$ and thus Δ is a pretopology.

It is left to prove that the functions defined by the above formulas are mutually inverse.

Let cl_0 be a preclosure, let Δ be the pretopology induced by cl_0 by the formula (4), let cl_1 be the preclosure induced by Δ by the formula (3). Let's prove $\mathrm{cl}_1 = \mathrm{cl}_0$. Really,

$$\begin{aligned} x \in \mathrm{cl}_1(A) &\Leftrightarrow \\ \Delta(x) \not\asymp \uparrow^U A &\Leftrightarrow \\ \forall X \in \operatorname{up} \Delta(x) : X \cap A \neq \emptyset &\Leftrightarrow \\ \forall X \in \mathscr{P}U : (x \notin \mathrm{cl}_0(U \setminus X) \Rightarrow X \cap A \neq \emptyset) &\Leftrightarrow \\ \forall X' \in \mathscr{P}U : (x \notin \mathrm{cl}_0(X') \Rightarrow A \setminus X' \neq \emptyset) &\Leftrightarrow \\ \forall X' \in \mathscr{P}U : (A \setminus X' = \emptyset \Rightarrow x \in \mathrm{cl}_0(X')) &\Leftrightarrow \\ \forall X' \in \mathscr{P}U : (A \subseteq X' \Rightarrow x \in \mathrm{cl}_0(X')) &\Leftrightarrow \\ x \in \mathrm{cl}_0(A). \end{aligned}$$

So $\mathrm{cl}_1(A) = \mathrm{cl}_0(A)$.

Let now Δ_0 be a pretopology, let cl be the closure induced by Δ_0 by the formula (3), let Δ_1 be the pretopology induced by cl by the formula (4). Really

$$\begin{aligned} A \in \operatorname{up} \Delta_1(x) &\Leftrightarrow \\ x \notin \mathrm{cl}(U \setminus A) &\Leftrightarrow \\ \Delta_0(x) \asymp \uparrow^U (U \setminus A) &\Leftrightarrow \text{(proposition 551)} \\ \uparrow^U A \sqsupseteq \Delta_0(x) &\Leftrightarrow \\ A \in \operatorname{up} \Delta_0(x). \end{aligned}$$

So $\Delta_1(x) = \Delta_0(x)$.

That these functions are mutually inverse, is now proved. □

6.2.1. Pretopology induced by a metric. Every metric space induces a pretopology by the formula:

$$\Delta(x) = \prod^{\mathscr{P}U} \left\{ \frac{B_r(x)}{r \in \mathbb{R}, r > 0} \right\}.$$

Exercise 778. Show that it is a pretopology.

Proposition 779. The preclosure corresponding to this pretopology is the same as the preclosure of the metric space.

Proof. I denote the preclosure of the metric space as cl_M and the preclosure corresponding to our pretopology as cl_P. We need to show $\mathrm{cl}_P = \mathrm{cl}_M$. Really:

$$\begin{aligned} \mathrm{cl}_P(A) &= \\ \left\{ \frac{x \in U}{A \in \partial\Delta(x)} \right\} &= \\ \left\{ \frac{x \in U}{\forall \epsilon > 0 : B_\epsilon(x) \not\asymp A} \right\} &= \\ \left\{ \frac{y \in U}{\forall \epsilon > 0 \exists a \in A : d(y, a) < \epsilon} \right\} &= \\ \mathrm{cl}_M(A) \end{aligned}$$

for every set $A \in \mathscr{P}U$. □

6.3. Topological spaces

PROPOSITION 780. For the set of open sets of a metric space (U, d) it holds:

1°. Union of any (possibly infinite) number of open sets is an open set.
2°. Intersection of a finite number of open sets is an open set.
3°. U is an open set.

PROOF. Let S be a set of open sets. Let $a \in \bigcup S$. Then there exists $A \in S$ such that $a \in A$. Because A is open we have $B_r(a) \subseteq A$ for some $r > 0$. Consequently $B_r(a) \subseteq \bigcup S$ that is $\bigcup S$ is open.

Let $A_0, \dots, A_n$ be open sets. Let $a \in A_0 \cap \dots \cap A_n$ for some $n \in \mathbb{N}$. Then there exist r_i such that $B_{r_i}(a) \subseteq A_i$. So $B_r(a) \subseteq A_0 \cap \dots \cap A_n$ for $r = \min\{r_0, \dots, r_n\}$ that is $A_0 \cap \dots \cap A_n$ is open.

That U is an open set is obvious. □

The above proposition suggests the following definition:

DEFINITION 781. A *topology* on a set U is a collection $\mathcal{O}$ (called the set of *open sets*) of subsets of U such that:

1°. Union of any (possibly infinite) number of open sets is an open set.
2°. Intersection of a finite number of open sets is an open set.
3°. U is an open set.

The pair $(U, \mathcal{O})$ is called a *topological space*.

REMARK 782. From the above it is clear that every metric induces a topology.

PROPOSITION 783. Empty set is always open.

PROOF. Empty set is union of an empty set. □

DEFINITION 784. A *closed set* is a complement of an open set.

Topology can be equivalently expresses in terms of closed sets:

A *topology* on a set U is a collection (called the set of *closed sets*) of subsets of U such that:

1°. Intersection of any (possibly infinite) number of closed sets is a closed set.
2°. Union of a finite number of closed sets is a closed set.
3°. $\emptyset$ is a closed set.

EXERCISE 785. Show that the definitions using open and closed sets are equivalent.

6.3.1. Relationships between pretopologies and topologies.

6.3.1.1. *Topological space induced by preclosure space.* Having a preclosure space (U, cl) we define a topological space whose closed sets are such sets $A \in \mathscr{P}U$ that $\mathrm{cl}(A) = A$.

PROPOSITION 786. This really defines a topology.

PROOF. Let S be a set of closed sets. First, we need to prove that $\bigcap S$ is a closed set. We have $\mathrm{cl}(\bigcap S) \subseteq A$ for every $A \in S$. Thus $\mathrm{cl}(\bigcap S) \subseteq \bigcap S$ and consequently $\mathrm{cl}(\bigcap S) = \bigcap S$. So $\bigcap S$ is a closed set.

Let now $A_0, \dots, A_n$ be closed sets, then

$$\mathrm{cl}(A_0 \cup \dots \cup A_n) = \mathrm{cl}(A_0) \cup \dots \cup \mathrm{cl}(A_n) = A_0 \cup \dots \cup A_n$$

that is $A_0 \cup \dots \cup A_n$ is a closed set.

That $\emptyset$ is a closed set is obvious. □

Having a pretopological space (U, Δ) we define a topological space whose open sets are

$$\left\{ \frac{X \in \mathscr{P}U}{\forall x \in X : X \in \operatorname{up} \Delta(x)} \right\}.$$

PROPOSITION 787. This really defines a topology.

PROOF. Let set $S \subseteq \left\{ \frac{X \in \mathscr{P}U}{\forall x \in X : X \in \operatorname{up} \Delta(x)} \right\}$. Then $\forall X \in S \forall x \in X : X \in \operatorname{up} \Delta(x)$. Thus

$$\forall x \in \bigcup S \exists X \in S : X \in \operatorname{up} \Delta(x)$$

and so $\forall x \in \bigcup S : \bigcup S \in \operatorname{up} \Delta(x)$. So $\bigcup S$ is an open set.

Let now $A_0, \dots, A_n \in \left\{ \frac{X \in \mathscr{P}U}{\forall x \in X : X \in \operatorname{up} \Delta(x)} \right\}$ for $n \in \mathbb{N}$. Then $\forall x \in A_i : A_i \in \operatorname{up} \Delta(x)$ and so

$$\forall x \in A_0 \cap \dots \cap A_n : A_i \in \operatorname{up} \Delta(x);$$

thus $\forall x \in A_0 \cap \dots \cap A_n : A_0 \cap \dots \cap A_n \in \operatorname{up} \Delta(x)$. So $A_0 \cap \dots \cap A_n \in \left\{ \frac{X \in \mathscr{P}U}{\forall x \in X : X \in \operatorname{up} \Delta(x)} \right\}$.

That U is an open set is obvious. □

PROPOSITION 788. Topology τ defined by a pretopology and topology ρ defined by the corresponding preclosure, are the same.

PROOF. Let $A \in \mathscr{P}U$.

A is ρ-closed $\Leftrightarrow \operatorname{cl}(A) = A \Leftrightarrow \operatorname{cl}(A) \subseteq A \Leftrightarrow \forall x \in U : (A \in \partial\Delta(x) \Rightarrow x \in A)$;

$$\begin{aligned} A \text{ is } \tau\text{-open} &\Leftrightarrow \\ \forall x \in A : A \in \operatorname{up} \Delta(x) &\Leftrightarrow \\ \forall x \in U : (x \in A \Rightarrow A \in \operatorname{up} \Delta(x)) &\Leftrightarrow \\ \forall x \in U : (x \notin U \setminus A \Rightarrow U \setminus A \notin \partial\Delta(x)). \end{aligned}$$

So ρ-closed and τ-open sets are complements of each other. It follows $\rho = \tau$. □

6.3.1.2. *Preclosure space induced by topological space.* We define a preclosure and a pretopology induced by a topology and then show these two are equivalent.

Having a topological space we define a preclosure space by the formula

$$\operatorname{cl}(A) = \bigcap \left\{ \frac{X \in \mathscr{P}U}{X \text{ is a closed set}, X \supseteq A} \right\}.$$

PROPOSITION 789. It is really a preclosure.

PROOF. $\operatorname{cl}(\emptyset) = \emptyset$ because $\emptyset$ is a closed set. $\operatorname{cl}(A) \supseteq A$ is obvious.

$$\begin{aligned} \operatorname{cl}(A \cup B) &= \\ \bigcap \left\{ \frac{X \in \mathscr{P}U}{X \text{ is a closed set}, X \supseteq A \cup B} \right\} &= \\ \bigcap \left\{ \frac{X_1 \cup X_2}{X_1, X_2 \in \mathscr{P}U \text{ are closed sets}, X_1 \supseteq A, X_2 \supseteq B} \right\} &= \\ \bigcap \left\{ \frac{X_1 \in \mathscr{P}U}{X_1 \text{ is a closed set}, X_1 \supseteq A} \right\} \cup \bigcap \left\{ \frac{X_2 \in \mathscr{P}U}{X_2 \text{ is a closed set}, X_2 \supseteq B} \right\} &= \\ \operatorname{cl}(A) \cup \operatorname{cl}(B). \end{aligned}$$

Thus cl is a preclosure. □

Or: $\Delta(x) = \bigsqcap^{\mathscr{F}} \left\{ \frac{X \in \mathcal{O}}{x \in X} \right\}$.

It is trivially a pretopology (used the fact that $U \in \mathcal{O}$).

PROPOSITION 790. The preclosure and the pretopology defined in this section above correspond to each other (by the formulas from theorem 777).

PROOF. We need to prove $\mathrm{cl}(A) = \left\{ \frac{x \in U}{\Delta(x) \not\asymp \uparrow^U A} \right\}$, that is

$$\bigcap \left\{ \frac{X \in \mathscr{P}U}{X \text{ is a closed set}, X \supseteq A} \right\} = \left\{ \frac{x \in U}{\prod^{\mathscr{F}U} \left\{ \frac{X \in \mathcal{O}}{x \in X} \right\} \not\asymp \uparrow^U A} \right\}.$$

Equivalently transforming it, we get:

$$\bigcap \left\{ \frac{X \in \mathscr{P}U}{X \text{ is a closed set}, X \supseteq A} \right\} = \left\{ \frac{x \in U}{\forall X \in \mathcal{O} : (x \in X \Rightarrow \uparrow^U X \not\asymp \uparrow^U A)} \right\};$$
$$\bigcap \left\{ \frac{X \in \mathscr{P}U}{X \text{ is a closed set}, X \supseteq A} \right\} = \left\{ \frac{x \in U}{\forall X \in \mathcal{O} : (x \in X \Rightarrow X \not\asymp A)} \right\}.$$

We have

$$\begin{aligned} x \in \bigcap \left\{ \frac{X \in \mathscr{P}U}{X \text{ is a closed set}, X \supseteq A} \right\} &\Leftrightarrow \\ \forall X \in \mathscr{P}U : (X \text{ is a closed set} \wedge X \supseteq A \Rightarrow x \in X) &\Leftrightarrow \\ \forall X' \in \mathcal{O} : (U \setminus X' \supseteq A \Rightarrow x \in U \setminus X') &\Leftrightarrow \\ \forall X' \in \mathcal{O} : (X' \asymp A \Rightarrow x \notin X') &\Leftrightarrow \\ \forall X \in \mathcal{O} : (x \in X \Rightarrow X \not\asymp A). \end{aligned}$$

So our equivalence holds. □

PROPOSITION 791. If τ is the topology induced by pretopology π, in turn induced by topology ρ, then $\tau = \rho$.

PROOF. The set of closed sets of τ is

$$\begin{aligned} \left\{ \frac{A \in \mathscr{P}U}{\mathrm{cl}_\pi(A) = A} \right\} &= \\ \left\{ \frac{A \in \mathscr{P}U}{\bigcap \left\{ \frac{X \in \mathscr{P}U}{X \text{ is a closed set in } \rho, X \supseteq A} \right\} = A} \right\} &= \\ \left\{ \frac{A \in \mathscr{P}U}{A \text{ is a closed set in } \rho} \right\} \end{aligned}$$

(taken into account that intersecting closed sets is a closed set). □

DEFINITION 792. Idempotent closures are called *Kuratowski closures*.

THEOREM 793. The above defined correspondences between topologies and pretopologies, restricted to Kuratowski closures, is a bijection.

PROOF. Taking into account the above proposition, it's enough to prove that:

If τ is the pretopology induced by topology π, in turn induced by a Kuratowski closure ρ, then $\tau = \rho$.

$$\begin{aligned}
\mathrm{cl}_\tau(A) &= \\
\bigcap\left\{\frac{X \in \mathscr{P}U}{X \text{ is a closed set in } \pi, X \supseteq A}\right\} &= \\
\bigcap\left\{\frac{X \in \mathscr{P}U}{\mathrm{cl}_\rho(X) = X, X \supseteq A}\right\} &= \\
\bigcap\left\{\frac{\mathrm{cl}_\rho(X)}{X \in \mathscr{P}U, \mathrm{cl}_\rho(X) = X, X \supseteq \mathrm{cl}_\rho(A)}\right\} &= \\
\bigcap\left\{\frac{\mathrm{cl}_\rho(\mathrm{cl}_\rho(X))}{X = A}\right\} &= \\
\mathrm{cl}_\rho(\mathrm{cl}_\rho(A)) &= \\
\mathrm{cl}_\rho(A).
\end{aligned}$$

□

6.3.1.3. *Topology induced by a metric.*

Definition 794. Every metric space induces a topology in this way: A set X is open iff

$$\forall x \in X \exists \epsilon > 0 : B_r(x) \subseteq X.$$

Exercise 795. Prove it is really a topology and this topology is the same as the topology, induced by the pretopology, in turn induced by our metric space.

6.4. Proximity spaces

Let (U, d) be metric space. We will define *distance* between sets $A, B \in \mathscr{P}U$ by the formula

$$d(A, B) = \inf\left\{\frac{d(a, b)}{a \in A, b \in B}\right\}.$$

(Here "inf" denotes infimum on the real line.)

Definition 796. Sets $A, B \in \mathscr{P}U$ are *near* (denoted $A \,\delta\, B$) iff $d(A, B) = 0$.

δ defined in this way (for a metric space) is an example of proximity as defined below.

Definition 797. A *proximity space* is a set (U, δ) conforming to the following axioms (for every $A, B, C \in \mathscr{P}U$):

1°. $A \cap B \neq \emptyset \Rightarrow A \,\delta\, B$;
2°. if $A \,\delta\, B$ then $A \neq \emptyset$ and $B \neq \emptyset$;
3°. $A \,\delta\, B \Rightarrow B \,\delta\, A$ (*symmetry*);
4°. $(A \cup B) \,\delta\, C \Leftrightarrow A \,\delta\, C \vee B \,\delta\, C$;
5°. $C \,\delta\, (A \cup B) \Leftrightarrow C \,\delta\, A \vee C \,\delta\, B$;
6°. $A \,\bar{\delta}\, B$ implies existence of $P, Q \in \mathscr{P}U$ with $A \,\bar{\delta}\, P$, $B \,\bar{\delta}\, Q$ and $P \cup Q = U$.

Exercise 798. Show that proximity generated by a metric space is really a proximity (conforms to the above axioms).

Definition 799. *Quasi-proximity* is defined as the above but without the symmetry axiom.

Definition 800. Closure is generated by a proximity by the following formula:

$$\mathrm{cl}(A) = \left\{\frac{a \in U}{\{a\} \,\delta\, A}\right\}.$$

PROPOSITION 801. Every closure generated by a proximity is a Kuratowski closure.

PROOF. First prove it is a preclosure. $\operatorname{cl}(\emptyset) = \emptyset$ is obvious. $\operatorname{cl}(A) \supseteq A$ is obvious.

$$\begin{aligned}
\operatorname{cl}(A \cup B) &= \\
\left\{ \frac{a \in U}{\{a\} \ \delta \ A \cup B} \right\} &= \\
\left\{ \frac{a \in U}{\{a\} \ \delta \ A \vee \{a\} \ \delta \ B} \right\} &= \\
\left\{ \frac{a \in U}{\{a\} \ \delta \ A} \right\} \cup \left\{ \frac{a \in U}{\{a\} \ \delta \ B} \right\} &= \\
\operatorname{cl}(A) \cup \operatorname{cl}(B).
\end{aligned}$$

It is remained to prove that cl is idempotent, that is $\operatorname{cl}(\operatorname{cl}(A)) = \operatorname{cl}(A)$. It is enough to show $\operatorname{cl}(\operatorname{cl}(A)) \subseteq \operatorname{cl}(A)$ that is if $x \notin \operatorname{cl}(A)$ then $x \notin \operatorname{cl}(\operatorname{cl}(A))$.

If $x \notin \operatorname{cl}(A)$ then $\{x\} \ \bar{\delta} \ A$. So there are $P, Q \in \mathscr{P}U$ such that $\{x\} \ \bar{\delta} \ P$, $A \ \bar{\delta} \ Q$, $P \cup Q = U$. Then $U \setminus Q \subseteq P$, so $\{x\} \ \bar{\delta} \ U \setminus Q$ and hence $x \in Q$. Hence $U \setminus \operatorname{cl}(A) \subseteq Q$, and so $\operatorname{cl}(A) \subseteq U \setminus Q \subseteq P$. Consequently $\{x\} \ \bar{\delta} \ \operatorname{cl}(A)$ and hence $x \notin \operatorname{cl}(\operatorname{cl}(A))$. $\square$

6.5. Definition of uniform spaces

Here I will present the traditional definition of uniform spaces. Below in the chapter about reloids I will present a shortened and more algebraic (however a little less elementary) definition of uniform spaces.

DEFINITION 802. *Uniform space* is a pair (U, D) of a set U and filter $D \in \mathfrak{F}(U \times U)$ (called *uniformity* or the set of *entourages*) such that:

1°. If $F \in D$ then $\operatorname{id}_U \subseteq F$.
2°. If $F \in D$ then there exists $G \in D$ such that $G \circ G \subseteq F$.
3°. If $F \in D$ then $F^{-1} \in D$.