Certain categories are cartesian closed

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Abstract

I prove that the category of continuous maps between endofuncoids is cartesian closed. Whether the category of continuous maps between endoreloids is cartesian closed is yet an open problem.

This is a rough draft. There are errors!

Cartesian closed categories

Definition 1. A category is *cartesian closed* iff:

- It has finite products.
- For each objects A, B is given an object $\mathrm{MOR}(A;B)$ (*exponentiation*) and a morphism $\varepsilon^{\mathbf{Dig}}_{A,B}$: $\mathrm{MOR}(A;B)\times A\rightarrow B$.
- For each morphism $f\colon Z\times A\rightarrow B$ there is given a morphism (*exponential transpose*) $\sim f$: $Z\rightarrow\mathrm{MOR}(A;B)$.
- $\varepsilon\circ(\sim f\times 1_{\mathrm{A}})=f$.
- $\sim(\varepsilon\circ(g\times 1_A))=g$.

Our puspose is to prove (or disprove) that categories **Dig**, **Fcd**, and **Rld** are cartesian closed. Note that they have finite (and even infinite) products is already proved in http://www.mathematics21.org/binaries/product.pdf

Definitions of our categories

Categories **Dig**, **Fcd**, and **Rld** are respectively categories of:

1. discretely continuous maps between digraphs;
2. (proximally) continuous maps between endofuncoids;
3. (uniformly) continuous maps between endoreloids.

Definition 2. *Digraph* is an endomorphism of the category **Rel**.

Definition 3. Category **Dig** of digraphs is the category whose objects are digraphs and morphisms are discretely continuous maps between digraphs. That is morphisms from a digraph μ to a digraph ν are functions (or more precisely morphisms of **Set**) f such that $f\circ\mu\sqsubseteq\nu\circ f$ (or equivalently $\mu\sqsubseteq f^{-1}\circ\nu\circ f$ or equivalently $f\circ\mu\circ f^{-1}\sqsubseteq\nu$).

Remark 4. Category of digraphs is sometimes defined in an other (non equivalent) way, allowing multiple edges between two given vertices.

Definition 5. Category **Fcd** of continuous maps between endofuncoids is the category whose objects are endofuncoids and morphisms are proximally continuous maps between endofuncoids. That is morphisms from an endofuncoid μ to an endofuncoid ν are functions (or more precisely morphisms of **Set**) f such that $\uparrow^{\mathsf{FCD}} f \circ \mu \sqsubseteq \nu \circ \uparrow^{\mathsf{FCD}} f$ (or equivalently $\mu \sqsubseteq \uparrow^{\mathsf{FCD}} f^{-1} \circ \nu \circ \uparrow^{\mathsf{FCD}} f$ or equivalently $\uparrow^{\mathsf{FCD}} f \circ \mu \circ \uparrow^{\mathsf{FCD}} f^{-1} \sqsubseteq \nu$).

Definition 6. Category **Rld** of continuous maps between endoreloids is the category whose objects are endoreloids and morphisms are uniformly continuous maps between endoreloids. That is morphisms from an endoreloid μ to an endoreloid ν are functions (or more precisely morphisms of **Set**) f such that $\uparrow^{\mathsf{RLD}} f \circ \mu \sqsubseteq \nu \circ \uparrow^{\mathsf{RLD}} f$ (or equivalently $\mu \sqsubseteq \uparrow^{\mathsf{RLD}} f^{-1} \circ \nu \circ \uparrow^{\mathsf{RLD}} f$ or equivalently $\uparrow^{\mathsf{RLD}} f \circ \mu \circ \uparrow^{\mathsf{RLD}} f^{-1} \sqsubseteq \nu$).

Category of digraphs is cartesian closed

Category of digraphs is the simplest of our three categories and it is easy to demonstrate that it is cartesian closed. I demonstrate cartesian closedness of **Dig** mainly with the purpose to show a pattern similarly to which we may probably demonstrate our two other categories are cartesian closed.

Let G and H be graphs:

- $\operatorname{Ob} \operatorname{MOR}(G; H) = (\operatorname{Ob} H)^{\operatorname{Ob} G}$;
- $(f; g) \in \operatorname{GR} \operatorname{MOR}(G; H) \Leftrightarrow \forall (v; w) \in \operatorname{GR} G: (f(v); g(w)) \in \operatorname{GR} H$ for every $f, g \in \operatorname{Ob} \operatorname{MOR}(G; H) = (\operatorname{Ob} H)^{\operatorname{Ob} G}$;

$\operatorname{GR} 1_{\operatorname{MOR}(B;C)} = \operatorname{id}_{\operatorname{Ob} \operatorname{MOR}(B;C)} = \operatorname{id}_{(\operatorname{Ob} H)^{\operatorname{Ob} G}}$

Equivalently

$(f; g) \in \operatorname{GR} \operatorname{MOR}(G; H) \Leftrightarrow \forall (v; w) \in \operatorname{GR} G: g \circ \{(v; w)\} \circ f^{-1} \subseteq \operatorname{GR} H$

$(f; g) \in \operatorname{GR} \operatorname{MOR}(G; H) \Leftrightarrow g \circ (\operatorname{GR} G) \circ f^{-1} \subseteq \operatorname{GR} H$

$(f; g) \in \operatorname{GR} \operatorname{MOR}(G; H) \Leftrightarrow \langle f \times^{(C)} g \rangle \operatorname{GR} G \subseteq \operatorname{GR} H$

The transposition (the isomorphism) is uncurrying.

$\sim f = \lambda a \in Z \lambda y \in A: f(a; y)$ that is $(\sim f)(a)(y) = f(a; y)$.

$(-f)(a; y) = f(a)(y)$

If $f: A \times B \to C$ then $\sim f: A \to \operatorname{MOR}(B; C)$

Proposition 7. Transposition and its inverse are morphisms of **Dig**.

Proof. It follows from the equivalence $\sim f: A \to \operatorname{MOR}(B; C) \Leftrightarrow \forall x, y: (x A y \Rightarrow (\sim f)\, x\, (\operatorname{MOR}(B; C))\, (\sim f)\, y) \Leftrightarrow \forall x, y: (x A y \Rightarrow \forall (v; w) \in B: ((\sim f)\, x v; (\sim f)\, y w) \in C) \Leftrightarrow \forall x, y, v, w: (x A y \wedge v B w \Rightarrow ((\sim f)\, x v; (\sim f)\, y w) \in C) \Leftrightarrow \forall x, y, v, w: ((x; v)\, (A \times B)\, (y; w) \Rightarrow (f(x; v); f(y; w)) \in C) \Leftrightarrow f: A \times B \to C$. □

Evaluation $\varepsilon: \operatorname{MOR}(G; H) \times G \to H$ is defined by the formula:

Then evaluation is $\varepsilon_{B,C} = -(1_{\mathrm{MOR}(B;C)})$.

So $\varepsilon_{B,C}(p;q) = (-(1_{\mathrm{MOR}(B;C)}))(p;q) = (1_{\mathrm{MOR}(B;C)})(p)(q) = p(q)$.

Proposition 8. Evaluation is a morphism of **Dig**.

Proof. Because $\varepsilon_{B,C}(p;q) = -(1_{\mathrm{MOR}(B;C)})$. □

It remains to prove: [FIXME: $\varepsilon_{X,Y}$. What are X and Y?]

- $\varepsilon \circ (\sim f \times 1_{\mathrm{A}}) = f$;
- $\sim(\varepsilon \circ (g \times 1_{\mathrm{A}})) = g$.

Proof. $\varepsilon(\sim f \times 1_{\mathrm{A}})(a;p) = \varepsilon((\sim f)a;p) = (\sim f)ap = f(a;p)$. So $\varepsilon \circ (\sim f \times 1_{\mathrm{A}}) = f$.

$\sim(\varepsilon \circ (g \times 1_{\mathrm{A}}))(p)(q) = (\varepsilon \circ (g \times 1_{\mathrm{A}}))(p;q) = \varepsilon(g \times 1_{\mathrm{A}})(p;q) = \varepsilon(gp;q) = g(p)(q)$. So $\sim(\varepsilon \circ (g \times 1_{\mathrm{A}})) = g$. □

Exponentials in category Fcd

Define $\sim^{\mathbf{Fcd}} f = \uparrow^{\mathsf{FCD}} \sim^{\mathbf{Dig}} f$

Definition 9. A category is *cartesian closed* iff:

- $\varepsilon \circ (\sim f \times 1_{\mathrm{A}}) = f$.
- $\sim(\varepsilon \circ (g \times 1_A)) = g$.

But this follows from functoriality of $\uparrow^{\mathsf{FCD}}$.

??

Embed **Fcd** into **Dig** by the formulas:

$A \mapsto \lambda X \in \mathscr{P}\mathrm{Ob}\, A: \langle A \rangle X$

$f \mapsto \langle f \rangle$

Obviously this embedding (denote it T) is an injective (both on objects and morphisms) functor.

$\varepsilon^{\mathbf{Fcd}}_{A,B}(p \times q) = \langle p \rangle q$ [TODO: Should p and q be atomic?]

$\sim^{\mathbf{Rld}}$ is induced by $\sim^{\mathbf{Dig}}$.

Due its injectivity and functoriality, it is enough to prove:

1. binary products are preserved
2. $\varepsilon^{\mathbf{Dig}}_{\mathrm{T}A,\mathrm{T}B} = \mathrm{T}\varepsilon^{\mathbf{Fcd}}_{A,B}$
3. that $\sim^{\mathbf{Dig}} \mathrm{T} f = \mathrm{T} \sim^{\mathbf{Fcd}} f$ for every $f: A \to B$

$(\mathrm{T}\varepsilon^{\mathbf{Fcd}}_{A,B})(p \times q) = \langle \varepsilon^{\mathbf{Fcd}}_{A,B} \rangle (p \times q) = \langle p \rangle q$

$\varepsilon^{\mathbf{Dig}}_{\mathrm{T}A,\mathrm{T}B} X = (\mathrm{T}B)^{\mathrm{T}A} X = (\lambda Y \in \mathscr{P}\mathrm{Ob}\, B: \langle B \rangle Y)^{\lambda X \in \mathscr{P}\mathrm{Ob}\, A: \langle A \rangle X} X$

??

Due its injectivity and functoriality, it is enough to prove:

1. binary products are preserved
2. for every $\varepsilon^{\mathbf{Dig}}_{\mathrm{T}A,\mathrm{T}B}$ there exist $\varepsilon^{\mathbf{Fcd}}_{A,B}$ such that $\varepsilon^{\mathbf{Dig}}_{\mathrm{T}A,\mathrm{T}B} = \mathrm{T}\varepsilon^{\mathbf{Fcd}}_{A,B}$
3. for every $f: \mathrm{T}A \rightarrow \mathrm{T}B$ there exists $g: A \rightarrow B$ that $\sim^{\mathbf{Dig}} f = \mathrm{T}\sim^{\mathbf{Fcd}} g$

Consider $\varepsilon^{\mathbf{Dig}}_{\mathrm{T}A,\mathrm{T}B}$. Then $\varepsilon^{\mathbf{Dig}}_{\mathrm{T}A,\mathrm{T}B} X = (\mathrm{T}B)^{\mathrm{T}A} X = (\lambda X \in \mathscr{P}\operatorname{Ob} B: \langle B\rangle X)^{\lambda X \in \mathscr{P}\operatorname{Ob} A: \langle A\rangle X} X \in (\lambda X \in \mathscr{P}\operatorname{Ob} B: \langle B\rangle X)$ for as suitable X. Thus ?? $\varepsilon^{\mathbf{Dig}}_{\mathrm{T}A,\mathrm{T}B} 0 = 0$ and $\varepsilon^{\mathbf{Dig}}_{\mathrm{T}A,\mathrm{T}B}(I \cup J) = \varepsilon^{\mathbf{Dig}}_{\mathrm{T}A,\mathrm{T}B} I \cup \varepsilon^{\mathbf{Dig}}_{\mathrm{T}A,\mathrm{T}B} J$. Consequently $\varepsilon^{\mathbf{Fcd}}_{A,B}$ exists.

Consider $f: \mathrm{T}A \rightarrow \mathrm{T}B$.

??

Then $f \in (\mathrm{T}B)^{\mathrm{T}A}$ and $f \in \mathrm{C}(\mathrm{T}A; \mathrm{T}B)$.

$fX = ??$

$(\sim^{\mathbf{Dig}} f)(p; q) = f(p)(q) =$

Thus ??

??

Binary products are subatomic products and so are compatible with products of graphs.

A try to prove this directly:

Proposition 10. Transposition and its inverse are morphisms of **Fcd**.

Proof. ?? [TODO: Use below sets instead of ultrafilters.]

It follows from the equivalence (??is it an equivalence? the last step seems just an implication) $\sim f: A \rightarrow \mathrm{MOR}(B; C) \Leftrightarrow \forall x, y \in \operatorname{atoms}^{\mathfrak{F}}: (x\ [A]\ y \Rightarrow \langle \sim f\rangle\ x\ [\mathrm{MOR}(B; C)]\ \langle \sim f\rangle y) \Leftrightarrow \forall x, y \in \operatorname{atoms}^{\mathfrak{F}}: (x\ [A]\ y \Rightarrow \forall (v; w) \in \operatorname{atoms} B: (\langle \sim f\rangle x v \times^{\mathsf{FCD}} \langle \sim f\rangle y w) \in \operatorname{atoms} C) \Leftrightarrow \forall x, y, v, w: (x\ [A]\ y \wedge v\ [B]\ w \Rightarrow (\langle \sim f\rangle x v \times^{\mathsf{FCD}} \langle \sim f\rangle y w) \in \operatorname{atoms} C) \Leftrightarrow \forall x, y, v, w \in \operatorname{atoms}^{\mathfrak{F}}: (x \times^{\mathsf{RLD}} v\ [A \times B]\ y \times^{\mathsf{RLD}} w \Rightarrow (f(x; v); f(y; w)) \in C) \Leftrightarrow f: A \times B \rightarrow C$. □

Exponentials in category Rld

TODO