Filters on posets and generalizations

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Abstract

They are studied in details properties of filters on lattices, filters on posets, and certain generalizations thereof. Also it's done some more general lattice theory research. There are posed several open problems. Detailed study of filters is required for my ongoing research which will be published as "Algebraic General Topology" series.

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Contents

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1. Preface

This article is intended to collect in one document the known properties of filters on posets (and some generalizations thereof, namely "filtrators" defined below).

It seems that until now were published no reference on the theory of filters. This text is to fill the gap.

This text will also serve as the reference base for my further articles. This text provides a definitive place to refer as to the collection of theorems about filters.

Detailed study of filters is required for my ongoing research which will be published as "Algebraic General Topology" series.

In place of studying filters in this article are instead researched what the author calls "filter objects". Filter objects are basically the lattice of filters ordered reverse to set inclusion, with principal filters equated with the poset element which generates them. (See below for formal definition of "filter objects".)

Although our primary interest are properties of filters on a set, in this work are instead researched the more general theory of "filtrators" (see below).

This article also contains some original research:

- filtrators;
- straight maps and separation subsets;
- other minor results, such as the theory of free stars.

2. Notation and basic results

We denote $\mathcal{P}S$ the set of all subsets of a set S.

$\langle f\rangle X \overset{\text{def}}{=} \{fx \mid x \in X\}$ for any set X and function f.

2.1. Intersecting and joining elements

Let $\mathfrak{A}$ be a poset.

Definition 1 *I will call elements a and b of $\mathfrak{A}$* ***intersecting*** *and denote $a \not\asymp b$ when exists not least element c such that $c \subseteq a \wedge c \subseteq b$.*

Definition 2 $a \asymp b \stackrel{\text{def}}{=} \neg(a \not\asymp b)$.

Obvious 1 *If $\mathfrak{A}$ is a meet-semilattice then $a \not\asymp b$ iff $a \cap b$ is non-least.*

Obvious 2 $a_0 \not\asymp b_0 \wedge a_1 \supseteq a_0 \wedge b_1 \supseteq b_0 \Rightarrow a_1 \not\asymp b_1$.

Definition 3 *I will call elements a and b of $\mathfrak{A}$* ***joining*** *and denote $a \equiv b$ when not exists not greatest element c such that $c \supseteq a \wedge c \supseteq b$.*

Definition 4 $a \not\equiv b \stackrel{\text{def}}{=} \neg(a \equiv b)$.

Obvious 3 *Intersecting is the dual of non-joining.*

Obvious 4 *If $\mathfrak{A}$ is a join-semilattice then $a \equiv b$ iff $a \cup b$ is its greatest element.*

Obvious 5 $a_0 \equiv b_0 \wedge a_1 \supseteq a_0 \wedge b_1 \supseteq b_0 \Rightarrow a_1 \equiv b_1$.

2.2. Atoms of a poset

Definition 5 *An* ***atom*** *of the poset is an element which has no non-least subelements.*

Remark 1 This definition is valid even for posets without least element.

I will denote $(\operatorname{atoms}^{\mathfrak{A}} a)$ or just $(\operatorname{atoms} a)$ the set of atoms contained in element a of a poset $\mathfrak{A}$.

Definition 6 *A poset $\mathfrak{A}$ is called* ***atomic*** *when $\operatorname{atoms} a \neq \emptyset$ for every non-least element $a \in \mathfrak{A}$.*

Definition 7 ***Atomistic poset*** *is such poset that $a = \bigcup \operatorname{atoms} a$ for every element a of this poset.*

Proposition 1 *Let $\mathfrak{A}$ be a poset. If a is an atom of $\mathfrak{A}$ and $B \in \mathfrak{A}$ then $a \subseteq B \Leftrightarrow a \not\asymp B$.*

Proof

$\Rightarrow$ $a \subseteq B \Rightarrow a \subseteq a \wedge a \subseteq B$, thus $a \not\asymp B$ because a is not least.

$\Leftarrow$ $a \not\asymp B$ implies existence of non-least element x such that $x \subseteq B$ and $x \subseteq a$. Because a is an atom, we have $x = a$. So $a \subseteq B$.

□

Theorem 1 *If $\mathfrak{A}$ is a distributive lattice then*

$$\operatorname{atoms}(a \cup b) = \operatorname{atoms} a \cup \operatorname{atoms} b$$

for every $a, b \in \mathfrak{A}$.

Proof For any atomic element c

$$\begin{aligned} c \in \operatorname{atoms}(a \cup b) &\Leftrightarrow \\ c \cap (a \cup b) \text{ is not least} &\Leftrightarrow \\ (c \cap a) \cup (c \cap b) \text{ is not least} &\Leftrightarrow \\ c \cap a \text{ is not least or } c \cap b \text{ is not least} &\Leftrightarrow \\ c \in \operatorname{atoms} a \vee c \in \operatorname{atoms} b. & \end{aligned}$$

□

Theorem 2 $\operatorname{atoms} \bigcap S = \bigcap \langle \operatorname{atoms} \rangle S$ *whenever $\bigcap S$ is defined for every $S \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a poset.*

Proof For any atom c

$$\begin{aligned} c \in \operatorname{atoms} \bigcap S &\Leftrightarrow \\ c \subseteq \bigcap S &\Leftrightarrow \\ \forall a \in S : c \subseteq a &\Leftrightarrow \\ \forall a \in S : c \in \operatorname{atoms} a &\Leftrightarrow \\ c \in \bigcap \langle \operatorname{atoms} \rangle S. & \end{aligned}$$

□

Corollary 1 $\operatorname{atoms}(a \cap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$ *for arbitrary meet-semilattice.*

Theorem 3 *A complete boolean lattice is atomic iff it is atomistic.*

Proof

$\Leftarrow$ Obvious.

$\Rightarrow$ Let $\mathfrak{A}$ be an atomic boolean lattice. Let $a \in \mathfrak{A}$. Suppose $b = \bigcup \operatorname{atoms} a \subset a$. If $x \in \operatorname{atoms}(a \setminus b)$ then $x \subseteq a \setminus b$ and so $x \subseteq a$ and hence $x \subseteq b$. But we have $x = x \cap b \subseteq (a \setminus b) \cap b = 0$ what contradicts to our supposition.

□

2.3. Difference and complement

Definition 8 *Let $\mathfrak{A}$ be a distributive lattice with least element 0. The* ***difference*** *(denoted $a \setminus b$) of elements a and b is such $c \in \mathfrak{A}$ that $b \cap c = 0$ and $a \cup b = b \cup c$. I will call b* ***substractive*** *from a when $a \setminus b$ exists.*

Theorem 4 *If $\mathfrak{A}$ is a distributive lattice with least element 0, there exists no more than one difference of elements $a, b \in \mathfrak{A}$.*

Proof Let c and d are both differences $a \setminus b$. Then $b \cap c = b \cap d = 0$ and $a \cup b = b \cup c = b \cup d$. So

$$c = c \cap (b \cup c) = c \cap (b \cup d) = (c \cap b) \cup (c \cap d) = 0 \cup (c \cap d) = c \cap d.$$

Analogously, $d = d \cap c$. Consequently $c = c \cap d = d \cap c = d$. □

Definition 9 *I will call b* ***complementive*** *to a when there exists $c \in \mathfrak{A}$ such that $b \cap c = 0$ and $b \cup c = a$.*

Proposition 2 *b is complementive to a iff b is substractive from a and $b \subseteq a$.*

Proof

$\Leftarrow$ Obvious.

$\Rightarrow$ We deduce $b \subseteq a$ from $b \cup c = a$. Thus $a \cup b = a = b \cup c$.

□

Proposition 3 *If b is complementive to a then $(a \setminus b) \cup b = a$.*

Proof Because $b \subseteq a$ by the previous proposition. □

Definition 10 *Let $\mathfrak{A}$ be a bounded distributive lattice. The* ***complement*** *(denoted $\bar{a}$) of element $a \in \mathfrak{A}$ is such $b \in \mathfrak{A}$ that $a \cap b = 0$ and $a \cup b = 1$.*

Proposition 4 *If $\mathfrak{A}$ is a bounded distributive lattice then $\bar{a} = 1 \setminus a$.*

Proof $b = \bar{a} \Leftrightarrow b \cap a = 0 \wedge b \cup a = 1 \Leftrightarrow b \cap a = 0 \wedge 1 \cup a = a \cup b \Leftrightarrow b = 1 \setminus a$. □

Corollary 2 *If $\mathfrak{A}$ is a bounded distributive lattice then exists no more than one complement of an element $a \in \mathfrak{A}$.*

Definition 11 *An element of bounded distributive lattice is called* ***complemented*** *when its complement exists.*

Definition 12 *A distributive lattice is a* ***complemented lattice*** *iff every its element is complemented.*

Proposition 5 *For a distributive lattice $(a \setminus b) \setminus c = a \setminus (b \cup c)$ if $a \setminus b$ and $(a \setminus b) \setminus c$ are defined.*

Proof $((a \setminus b) \setminus c) \cap c = 0$; $((a \setminus b) \setminus c) \cup c = (a \setminus b) \cup c$; $(a \setminus b) \cap b = 0$; $(a \setminus b) \cup b = a \cup b$.

We need to prove $((a \setminus b) \setminus c) \cap (b \cup c) = 0$ and $((a \setminus b) \setminus c) \cup (b \cup c) = a \cup (b \cup c)$.

In fact,

$$\begin{aligned} ((a \setminus b) \setminus c) \cap (b \cup c) &= \\ (((a \setminus b) \setminus c) \cap b) \cup (((a \setminus b) \setminus c) \cap c) &= \\ (((a \setminus b) \setminus c) \cap b) \cup 0 &= \\ ((a \setminus b) \setminus c) \cap b &\subseteq \\ (a \setminus b) \cap b &= 0, \end{aligned}$$

so $((a \setminus b) \setminus c) \cap (b \cup c) = 0$;

$$\begin{aligned} ((a \setminus b) \setminus c) \cup (b \cup c) &= \\ (((a \setminus b) \setminus c) \cup c) \cup b &= \\ (a \setminus b) \cup c \cup b &= \\ ((a \setminus b) \cup b) \cup c &= \\ a \cup b \cup c. \end{aligned}$$

□

2.4. Center of a lattice

Definition 13 *The **center** $Z(\mathfrak{A})$ of a bounded distributive lattice $\mathfrak{A}$ is the set of its complemented elements.*

Remark 2 For definition of center of non-distributive lattices see [3].

Remark 3 In [9] the word center and the notation $Z(\mathfrak{A})$ is used in a different sense.

Definition 14 *A complete lattice $\mathfrak{A}$ is **join infinite distributive** when $x \cap \bigcup S = \bigcup \langle x\cap \rangle S$; complete lattice is **meet infinite distributive** when $x \cup \bigcap S = \bigcap \langle x\cup \rangle S$ for all $x \in \mathfrak{A}$ and $S \in \mathcal{P}\mathfrak{A}$.*

Definition 15 ***Infinitely distributive complete lattice** is a complete lattice which is both join infinite distributive and meet infinite distributive.*

Definition 16 *A sublattice K of a complete lattice L is a closed sublattice of L if K contains the meet and the join of any its nonempty subset.*

Theorem 5 *Center of a infinitely distributive lattice is its closed sublattice.*

Proof See [6]. □

Remark 4 See [7] for a more strong result.

Theorem 6 *The center of a bounded distributive lattice constitutes its sublattice.*

Proof Let $\mathfrak{A}$ be a bounded distributive lattice and $Z(\mathfrak{A})$ is its center. Let $a, b \in Z(\mathfrak{A})$. Consequently $\bar{a}, \bar{b} \in Z(\mathfrak{A})$. Then $\bar{a} \cup \bar{b}$ is the complement of $a \cap b$ because

$$(a \cap b) \cap (\bar{a} \cup \bar{b}) = (a \cap b \cap \bar{a}) \cup (a \cap b \cap \bar{b}) = 0 \cup 0 = 0 \quad \text{and}$$
$$(a \cap b) \cup (\bar{a} \cup \bar{b}) = (a \cup \bar{a} \cup \bar{b}) \cap (b \cup \bar{a} \cup \bar{b}) = 1 \cap 1 = 1.$$

So $a \cap b$ is complemented, analogously $a \cup b$ is complemented. □

Theorem 7 *The center of a bounded distributive lattice constitutes a boolean lattice.*

Proof Because it is a distributive complemented lattice. □

2.5. Galois connections

See [1] and [5] for more detailed treatment of Galois connections.

Definition 17 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two posets. A* ***Galois connection*** *between $\mathfrak{A}$ and $\mathfrak{B}$ is a pair of functions $f = (f^*; f_*)$ with $f^* : \mathfrak{A} \rightarrow \mathfrak{B}$ and $f_* : \mathfrak{B} \rightarrow \mathfrak{A}$ such that:*

$$\forall x \in \mathfrak{A}, y \in \mathfrak{B} : (f^* x \subseteq^{\mathfrak{B}} y \Leftrightarrow x \subseteq^{\mathfrak{A}} f_* y).$$

f_* is called ***upper adjoint*** *of f^* and f^* is called* ***lower adjoint*** *of f_*.*

Theorem 8 *A pair $(f^*; f_*)$ of functions $f^* : \mathfrak{A} \rightarrow \mathfrak{B}$ and $f_* : \mathfrak{B} \rightarrow \mathfrak{A}$ is a Galois connection iff both of the following:*

1. f^* and f_* are monotone.

2. $x \subseteq^{\mathfrak{A}} f_* f^* x$ and $f^* f_* y \subseteq^{\mathfrak{B}} y$ for every $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$.

Proof

$\Rightarrow$ 2. $x \subseteq^{\mathfrak{A}} f_* f^* x$ since $f^* x \subseteq^{\mathfrak{B}} f^* x$; $f^* f_* y \subseteq^{\mathfrak{B}} y$ since $f_* y \subseteq^{\mathfrak{A}} f_* y$.

1. Let $a, b \in \mathfrak{A}$ and $a \subseteq^{\mathfrak{A}} b$. Then $a \subseteq^{\mathfrak{A}} b \subseteq^{\mathfrak{A}} f_* f^* b$. So by definition $f^* a \subseteq f^* b$ that is f^* is monotone. Analogously f_* is monotone.

$\Leftarrow$ $f^* x \subseteq^{\mathfrak{B}} y \Rightarrow f_* f^* x \subseteq^{\mathfrak{A}} f_* y \Rightarrow x \subseteq^{\mathfrak{A}} f_* y$. The other direction is analogous.

□

Theorem 9

1. $f^* \circ f_* \circ f^* = f^*$.

2. $f_* \circ f^* \circ f_* = f_*$.

Proof

1. Let $x \in \mathfrak{A}$. We have $x \sqsubseteq^{\mathfrak{A}} f_* f^* x$; consequently $f^* x \sqsubseteq^{\mathfrak{B}} f^* f_* f^* x$. On the other hand, $f^* f_* f^* x \sqsubseteq^{\mathfrak{B}} f^* x$. So $f^* f_* f^* x = f^* x$.

2. Analogously.

□

Proposition 6 *$f^* \circ f_*$ and $f_* \circ f^*$ are idempotent.*

Proof $f^* \circ f_*$ is idempotent because $f^* f_* f^* f_* y = f^* f_* y$. $f_* \circ f^*$ is similar. □

Theorem 10 *Each of two adjoints is uniquely determined by the other.*

Proof Let p and q be both upper adjoints of f. We have for all $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$:

$$x \sqsubseteq p(y) \Leftrightarrow f(x) \sqsubseteq y \Leftrightarrow x \sqsubseteq q(y).$$

For $x = p(y)$ we obtain $p(y) \sqsubseteq q(y)$ and for $x = q(y)$ we obtain $q(y) \sqsubseteq p(y)$. So $p(y) = q(y)$. □

Theorem 11 *Let f be a function from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.*

1. Both:

   1. If f is monotone and $g(b) = \max\{x \in \mathfrak{A} \mid fx \sqsubseteq b\}$ is defined for every $b \in \mathfrak{B}$ then g is the upper adjoint of f.

   2. If $g \colon \mathfrak{B} \to \mathfrak{A}$ is the upper adjoint of f then $g(b) = \max\{x \in \mathfrak{A} \mid fx \sqsubseteq b\}$ for every $b \in \mathfrak{B}$.

2. Both:

   1. If f is monotone and $g(b) = \min\{x \in \mathfrak{A} \mid fx \sqsupseteq b\}$ is defined for every $b \in \mathfrak{B}$ then g is the lower adjoint of f.

   2. If $g \colon \mathfrak{B} \to \mathfrak{A}$ is the lower adjoint of f then $g(b) = \min\{x \in \mathfrak{A} \mid fx \sqsupseteq b\}$ for every $b \in \mathfrak{B}$.

Proof We will prove only the first as the second is its dual.

1. Let $g(b) = \max\{x \in \mathfrak{A} \mid fx \subseteq b\}$ for every $b \in \mathfrak{B}$. Then

$$x \subseteq gy \Leftrightarrow x \subseteq \max\{x \in \mathfrak{A} \mid fx \subseteq y\} \Rightarrow fx \subseteq y$$

(because f is monotone) and

$$x \subseteq gy \Leftrightarrow x \subseteq \max\{x \in \mathfrak{A} \mid fx \subseteq y\} \Leftarrow fx \subseteq y.$$

So $fx \subseteq y \Leftrightarrow x \subseteq gy$ that is f is the lower adjoint of g.

2. We have

$$g(b) = \max\{x \in \mathfrak{A} \mid fx \subseteq b\} \quad \Leftrightarrow$$
$$fgb \subseteq b \wedge \forall x \in \mathfrak{A} : (fx \subseteq b \Rightarrow x \subseteq gb)$$

what is true by properties of adjoints.

$\square$

Theorem 12 *Let f be a function from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.*

1. If f is an upper adjoint, f preserves all existing infima in $\mathfrak{A}$.

2. If $\mathfrak{A}$ is a complete lattice and f preserves all infima, then f is an upper adjoint of a function $\mathfrak{B} \to \mathfrak{A}$.

3. If f is a lower adjoint, f preserves all existing suprema in $\mathfrak{A}$.

4. If $\mathfrak{A}$ is a complete lattice and f preserves all suprema, then f is a lower adjoint of a function $\mathfrak{B} \to \mathfrak{A}$.

Proof We will prove only first two items because the rest items are similar.

1. Let $S \in \mathcal{P}\mathfrak{A}$ and $\bigcap S$ exists. $f \bigcap S$ is a lower bound for $\langle f \rangle S$ because f is order-preserving. If a is a lower bound for $\langle f \rangle S$ then $\forall x \in S : a \subseteq fx$ that is $\forall x \in S : x \subseteq ga$ where g is the lower adjoint of f. Thus $\bigcap S \subseteq ga$ and hence $f \bigcap S \subseteq a$. So $f \bigcap S$ is the greatest lower bound for $\langle f \rangle S$.

2. Let $\mathfrak{A}$ be a complete lattice and f preserves all infima. Let $g(a) = \bigcap\{x \in \mathfrak{A} \mid fx \supseteq a\}$. Since f preserves infima, we have

$$f(g(a)) = \bigcap\{f(x) \mid x \in \mathfrak{A}, f(x) \supseteq a\} \supseteq a.$$

$g(f(b)) = \bigcap\{x \in \mathfrak{A} \mid fx \supseteq fb\} \subseteq b$.

Obviously f is monotone and thus g is also monotone.

So f is the upper adjoint of g.

$\square$

Corollary 3 *Let f be a function from a complete lattice $\mathfrak{A}$ to a poset $\mathfrak{B}$. Then:*

1. f is an upper adjoint of a function $\mathfrak{B} \to \mathfrak{A}$ iff f preserves all infima in $\mathfrak{A}$.

2. f is a lower adjoint of a function $\mathfrak{B} \to \mathfrak{A}$ iff f preserves all suprema in $\mathfrak{A}$.

Definition 18 *Let $\mathfrak{A}$ be a poset. Let $a \in \mathfrak{A}$.* ***Pseudocomplement*** *of a is*

$$\max \{c \in \mathfrak{A} \mid c \asymp a\}.$$

If z is pseudocomplement of a we will denote $z = a^*$.

Definition 19 *Let $\mathfrak{A}$ be a poset. Let $a \in \mathfrak{A}$.* ***Dual pseudocomplement*** *of a is*

$$\min \{c \in \mathfrak{A} \mid c \equiv a\}.$$

If z is dual pseudocomplement of a we will denote $z = a^+$.

Definition 20 *Let $\mathfrak{A}$ be a join-semilattice. Let $a, b \in \mathfrak{A}$.* ***Pseudodifference*** *of a and b is*

$$\min \{z \in \mathfrak{A} \mid a \subseteq b \cup z\}.$$

If z is a pseudodifference of a and b we will denote $z = a \setminus^* b$.

Remark 5 I do not require that a^* is undefined if there are no pseudocomplement of a and likewise for dual pseudocomplement and pseudodifference. In fact below I will define quasicomplement, dual quasicomplement, and quasidifference which will generalize pseudo-* counterparts. I will denote a^* the more general case of quasicomplement than of pseudocomplement, and likewise for other notation.

Obvious 6 *Dual pseudocomplement is the dual of pseudocomplement.*

Definition 21 ***Co-brouwerian lattice*** *is a lattice for which is defined pseudodifference of any two its elements.*

Proposition 7 *Every non-empty co-brouwerian lattice $\mathfrak{A}$ has least element.*

Proof Let a be an arbitrary lattice element. Then $a \setminus^* a = \min \{z \in \mathfrak{A} \mid a \subseteq a \cup z\} = \min \mathfrak{A}$. So $\min \mathfrak{A}$ exists. □

Definition 22 ***Co-Heyting lattice*** *is co-brouwerian lattice with greatest element.*

Theorem 13 *For a co-brouwerian lattice $a \cup -$ is an upper adjoint of $- \setminus^* a$ for every $a \in \mathfrak{A}$.*

Proof $g(b) = \min \{x \in \mathfrak{A} \mid a \cup x \supseteq b\} = b \setminus^* a$ exists for every $b \in \mathfrak{A}$ and thus is the lower adjoint of $a \cup -$. □

Corollary 4 $\forall a, x, y \in \mathfrak{A} : (x \setminus^* a \subseteq y \Leftrightarrow x \subseteq a \cup y)$ *for a co-brouwerian lattice.*

Definition 23 *Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a complete lattice. Quasidifference $a \setminus^* b$ is defined by the formula*

$$a \setminus^* b = \bigcap \{ z \in \mathfrak{A} \mid a \subseteq b \cup z \} .$$

Remark 6 The more detailed theory of quasidifference (as well as quasicomplement and dual quasicomplement) will be considered below.

Lemma 1 *$(a \setminus^* b) \cup b = a \cup b$ for elements a, b of a meet infinite distributive complete lattice.*

Proof

$$\begin{aligned} (a \setminus^* b) \cup b &= \\ \bigcap \{ z \in \mathfrak{A} \mid a \subseteq b \cup z \} \cup b &= \\ \bigcap \{ z \cup b \mid z \in \mathfrak{A}, a \subseteq b \cup z \} &= \\ \bigcap \{ t \in \mathfrak{A} \mid t \supseteq b, a \subseteq t \} &= \\ a \cup b. & \end{aligned}$$

□

Theorem 14 *The following are equivalent for a complete lattice $\mathfrak{A}$:*

1. *$\mathfrak{A}$ is meet infinite distributive.*
2. *$\mathfrak{A}$ is a co-brouwerian lattice.*
3. *$\mathfrak{A}$ is a co-Heyting lattice.*
4. *$a \cup -$ has lower adjoint for every $a \in \mathfrak{A}$.*

Proof

(2)⇔(3) Obvious (taking in account completeness of $\mathfrak{A}$).

(4)⇒(1) Let $- \setminus^* a$ be the lower adjoint of $a \cup -$. Let $S \in \mathcal{P}\mathfrak{A}$. For every $y \in S$ we have $y \supseteq (a \cup y) \setminus^* a$ by properties of Galois connections; consequently $y \supseteq (\bigcap \langle a \cup \rangle S) \setminus^* a$; $\bigcap S \supseteq (\bigcap \langle a \cup \rangle S) \setminus^* a$. So

$$a \cup \bigcap S \supseteq ((\bigcap \langle a \cup \rangle S) \setminus^* a) \cup a \supseteq \bigcap \langle a \cup \rangle S.$$

But $a \cup \bigcap S \subseteq \bigcap \langle a \cup \rangle S$ is obvious.

(1)⇒(2) Let $a \setminus^* b = \bigcap \{ z \in \mathfrak{A} \mid a \subseteq b \cup z \}$. To prove that $\mathfrak{A}$ is a co-brouwerian lattice is enough to prove that $a \subseteq b \cup (a \setminus^* b)$. But it follows from the lemma.

(2)⇒(4) $a \setminus^* b = \min\{z \in \mathfrak{A} \mid a \subseteq b \cup z\}$. So $a \cup -$ is an upper adjoint of $- \setminus^* a$.

(1)⇒(4) Because $a \cup -$ preserves all meets.

□

Corollary 5 *Co-brouwerian lattices are distributive.*

The following theorem is essentially borrowed from [8]:

Theorem 15 *A lattice $\mathfrak{A}$ with least element 0 is co-brouwerian with pseudodifference $\setminus^*$ iff $\setminus^*$ is a binary operation on $\mathfrak{A}$ satisfying the following identities:*

1. $a \setminus^* a = 0$;

2. $a \cup (b \setminus^* a) = a \cup b$;

3. $b \cup (b \setminus^* a) = b$;

4. $(b \cup c) \setminus^* a = (b \setminus^* a) \cup (c \setminus^* a)$.

Proof

⇐ We have

$$c \supseteq b \setminus^* a \Rightarrow c \cup a \supseteq a \cup (b \setminus^* a) = a \cup b \supseteq b;$$

$$c \cup a \supseteq b \Rightarrow c = c \cup (c \setminus^* a) \supseteq (a \setminus^* a) \cup (c \setminus^* a) = (a \cup c) \setminus^* a \supseteq b \setminus^* a.$$

So $c \supseteq b \setminus^* a \Leftrightarrow c \cup a \supseteq b$ that is $a \cup -$ is an upper adjoint of $- \setminus^* a$. By a theorem above our lattice is co-brouwerian. By an other theorem above $\setminus^*$ is a pseudodifference.

⇒ 1. Obvious.

2.

$$\begin{aligned} a \cup (b \setminus^* a) &= \\ a \cup \bigcap \{z \in \mathfrak{A} \mid b \subseteq a \cup z\} &= \\ \bigcap \{a \cup z \mid z \in \mathfrak{A}, b \subseteq a \cup z\} &= \\ a \cup b. \end{aligned}$$

3. $b \cup (b \setminus^* a) = b \cup \bigcap \{z \in \mathfrak{A} \mid b \subseteq a \cup z\} = \bigcap \{b \cup z \mid z \in \mathfrak{A}, b \subseteq a \cup z\} = b$.

4. Obviously $(b \cup c) \setminus^* a \supseteq b \setminus^* a$ and $(b \cup c) \setminus^* a \supseteq c \setminus^* a$, thus $(b \cup c) \setminus^* a \supseteq (b \setminus^* a) \cup (c \setminus^* a)$. We have

$$\begin{aligned} (b \setminus^* a) \cup (c \setminus^* a) \cup a &= \\ ((b \setminus^* a) \cup a) \cup ((c \setminus^* a) \cup a) &= \\ (b \cup a) \cup (c \cup a) &= \\ a \cup b \cup c &\supseteq \\ b \cup c. & \end{aligned}$$

From this by the definition of adjoints: $(b \setminus^* a) \cup (c \setminus^* a) \supseteq (b \cup c) \setminus^* a$.

□

Theorem 16 $(\bigcup S) \setminus^* a = \bigcup \{ x \setminus^* a \mid x \in S \}$ *for* $a \in \mathfrak{A}$ *and* $S \in \mathscr{P}\mathfrak{A}$ *where* $\mathfrak{A}$ *is a complete co-brouwerian lattice.*

Proof Because lower adjoint preserves all suprema. □

Theorem 17 $(a \setminus^* b) \setminus^* c = a \setminus^* (b \cup c)$ *for elements* a, b, c *of a complete co-brouwerian lattice.*

Proof $a \setminus^* b = \bigcap \{ z \in \mathfrak{A} \mid a \subseteq b \cup z \}$.
$(a \setminus^* b) \setminus^* c = \bigcap \{ z \in \mathfrak{A} \mid a \setminus^* b \subseteq c \cup z \}$.
$a \setminus^* (b \cup c) = \bigcap \{ z \in \mathfrak{A} \mid a \subseteq b \cup c \cup z \}$.
It's left to prove $a \setminus^* b \subseteq c \cup z \Leftrightarrow a \subseteq b \cup c \cup z$.
Let $a \setminus^* b \subseteq c \cup z$. Then $a \cup b \subseteq b \cup c \cup z$ by the lemma and consequently $a \subseteq b \cup c \cup z$.
Let $a \subseteq b \cup c \cup z$. Then $a \setminus^* b \subseteq (b \cup c \cup z) \setminus^* b \subseteq c \cup z$ by a theorem above. □

3. Straight maps and separation subsets

3.1. Straight maps

Definition 24 *Let* f *be a monotone map from a meet-semilattice* $\mathfrak{A}$ *to some poset* $\mathfrak{B}$. *I call* f *a* ***straight*** *map when*

$$\forall a, b \in \mathfrak{A} : (fa \subseteq fb \Rightarrow fa = f(a \cap b)).$$

Proposition 8 *The following statements are equivalent for a monotone map* f*:*

1. f *is a straight map.*

2. $\forall a, b \in \mathfrak{A} : (fa \subseteq fb \Rightarrow fa \subseteq f(a \cap b))$.

3. $\forall a, b \in \mathfrak{A} : (fa \subseteq fb \Rightarrow fa \not\supset f(a \cap b))$.

4. $\forall a, b \in \mathfrak{A} : (fa \supset f(a \cap b) \Rightarrow fa \nsubseteq fb)$.

Proof

(1)⇔(2)⇔(3) Due $fa \supseteq f(a \cap b)$.

(3)⇔(4) Obvious.

$\square$

Remark 7 The definition of straight map can be generalized for any poset $\mathfrak{A}$ by the formula

$$\forall a, b \in \mathfrak{A} : (fa \subseteq fb \Rightarrow \exists c \in \mathfrak{A} : (c \subseteq a \wedge c \subseteq b \wedge fa = fc)).$$

This generalization is not yet researched however.

Proposition 9 *Let f be a monotone map from a meet-semilattice $\mathfrak{A}$ to some poset $\mathfrak{B}$. If*

$$\forall a, b \in \mathfrak{A} : (f(a \cap b) = fa \cap fb)$$

then f is a straight map.

Proof Let $fa \subseteq fb$. Then $f(a \cap b) = fa \cap fb = fa$. $\square$

Proposition 10 *Let f be a monotone map from a meet-semilattice $\mathfrak{A}$ to some poset $\mathfrak{B}$. If*

$$\forall a, b \in \mathfrak{A} : (fa \subseteq fb \Rightarrow a \subseteq b)$$

then f is a straight map.

Proof $fa \subseteq fb \Rightarrow a \subseteq b \Rightarrow a = a \cap b \Rightarrow fa = f(a \cap b)$. $\square$

Theorem 18 *If f is a straight monotone map from a meet-semilattice $\mathfrak{A}$ then the following statements are equivalent:*

1. *f is an injection.*

2. $\forall a, b \in \mathfrak{A} : (fa \subseteq fb \Rightarrow a \subseteq b)$.

3. $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow fa \subset fb)$.

4. $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow fa \neq fb)$.

5. $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow fa \nsupseteq fb)$.

6. $\forall a, b \in \mathfrak{A} : (fa \subseteq fb \Rightarrow a \not\supset b)$.

Proof

(1)⇒(3) Let $a, b \in \mathfrak{A}$. Let $fa = fb \Rightarrow a = b$. Let $a \subset b$. $fa \neq fb$ because $a \neq b$. $fa \subseteq fb$ because $a \subseteq b$. So $fa \subset fb$.

(2)⇒(1) Let $a, b \in \mathfrak{A}$. Let $fa \subseteq fb \Rightarrow a \subseteq b$. Let $fa = fb$. Then $a \subseteq b \wedge b \subseteq a$ and consequently $a = b$.

(3)⇒(2) Let $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow fa \subset fb)$. Let $a \nsubseteq b$. Then $a \supset a \cap b$. So $fa \supset f(a \cap b)$. If $fa \subseteq fb$ then $fa \subseteq f(a \cap b)$ what is a contradiction.

(3)⇒(5)⇒(4) Obvious.

(4)⇒(3) Because $a \subset b \Rightarrow a \subseteq b \Rightarrow fa \subseteq fb$.

(5)⇔(6) Obvious.

□

3.2. Separation subsets and full stars

Definition 25 *$\partial_Y a = \{x \in Y \mid x \not\asymp a\}$ for an element a of a poset $\mathfrak{A}$ and $Y \in \mathcal{P}\mathfrak{A}$.*

Definition 26 ***Full star*** *of a is $\star a = \partial_{\mathfrak{A}} a$.*

Proposition 11 *If $\mathfrak{A}$ is a meet-semilattice, then $\star$ is a straight monotone map.*

Proof Monotonicity is obvious. Let $\star a \nsubseteq \star(a \cap b)$. Then it exists $x \in \star a$ such that $x \notin \star(a \cap b)$. So $x \cap a \notin \star b$ but $x \cap a \in \star a$ and consequently $\star a \nsubseteq \star b$. □

Definition 27 *A **separation subset** of a poset $\mathfrak{A}$ is such its subset Y that*

$$\forall a, b \in \mathfrak{A} : (\partial_Y a = \partial_Y b \Rightarrow a = b).$$

Definition 28 *I call **separable** such poset that $\star$ is an injection.*

Obvious 7 *A poset is separable iff it has separation subset.*

Definition 29 *A poset $\mathfrak{A}$ has **disjunction property of Wallman** iff for any $a, b \in \mathfrak{A}$ either $b \subseteq a$ or there exists a non-least element $c \subseteq b$ such that $a \asymp c$.*

Theorem 19 *For a meet-semilattice with least element the following statements are equivalent:*

1. *$\mathfrak{A}$ is separable.*
2. *$\forall a, b \in \mathfrak{A} : (\star a \subseteq \star b \Rightarrow a \subseteq b)$.*
3. *$\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow \star a \subset \star b)$.*
4. *$\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow \star a \neq \star b)$.*

5. $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow \star a \not\supseteq \star b)$*.*

6. $\forall a, b \in \mathfrak{A} : (\star a \subseteq \star b \Rightarrow a \not\supset b)$*.*

7. $\mathfrak{A}$ *conforms to Wallman's disjunction property.*

8. $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow \exists c \in \mathfrak{A} \setminus \{0\} : (c \asymp a \wedge c \subseteq b))$*.*

Proof

(1)⇔(2)⇔(3)⇔(4)⇔(5)⇔(6) By the above theorem.

(8)⇒(4) Let the property (8) holds. Let $a \subset b$. Then it exists element $c \subseteq b$ such that $c \neq 0$ and $c \cap a = 0$. But $c \cap b \neq 0$. So $\star a \neq \star b$.

(2)⇒(7) Let the property (2) holds. Let $a \nsubseteq b$. Then $\star a \nsubseteq \star b$ that is exists $c \in \star a$ such that $c \notin \star b$, in other words $c \cap a \neq 0$ and $c \cap b = 0$. Let $d = c \cap a$. Then $d \subseteq a$ and $d \neq 0$ and $d \cap b = 0$. So disjunction property of Wallman holds.

(7)⇒(8) Obvious.

(8)⇒(7) Let $b \nsubseteq a$. Then $a \cap b \subset b$ that is $a' \subset b$ where $a' = a \cap b$. Consequently $\exists c \in \mathfrak{A} \setminus \{0\} : (c \asymp a' \wedge c \subseteq b)$. We have $c \cap a = c \cap b \cap a = c \cap a'$. So $c \subseteq b$ and $c \cap a = 0$. Thus Wallman's disjunction property holds.

□

3.3. Atomically separable lattices

Proposition 12 *"*atoms*" is a straight monotone map (for any meet-semilattice).*

Proof Monotonicity is obvious. The rest follows from the formula

$$\operatorname{atoms}(a \cap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$$

(the corollary 1). □

Definition 30 *I will call* ***atomically separable*** *such a poset that "*atoms*" is an injection.*

Proposition 13 $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow \operatorname{atoms} a \subset \operatorname{atoms} b)$ *iff* $\mathfrak{A}$ *is atomically separable for a poset* $\mathfrak{A}$*.*

Proof

⇐ Obvious.

⇒ Let $a \neq b$ for example $a \nsubseteq b$. Then $a \cap b \subset a$; $\operatorname{atoms} a \supset \operatorname{atoms}(a \cap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$ and thus $\operatorname{atoms} a \neq \operatorname{atoms} b$.

Let $\operatorname{atoms} a \neq \operatorname{atoms} b$ for example $\operatorname{atoms} a \nsubseteq \operatorname{atoms} b$. Then $\operatorname{atoms}(a \cap b) = \operatorname{atoms} a \cap \operatorname{atoms} b \subset \operatorname{atoms} a$ and thus $a \cap b \subset a$ and so $a \nsubseteq b$ consequently $a \neq b$.

□

Proposition 14 *Any atomistic poset is atomically separable.*

Proof We need to prove that $\operatorname{atoms} a = \operatorname{atoms} b \Rightarrow a = b$. But it is obvious because

$$a = \bigcup \operatorname{atoms} a \quad \text{and} \quad b = \bigcup \operatorname{atoms} b.$$

□

Theorem 20 *If a lattice with least element is atomic and separable then it is atomistic.*

Proof Suppose the contrary that is $a \supset \bigcup \operatorname{atoms} a$. Then, because our lattice is separable, exists $c \in \mathfrak{A}$ such that $c \cap a \neq 0$ and $c \cap \bigcup \operatorname{atoms} a = 0$. There exist atom $d \subseteq c$ such that $d \subseteq c \cap a$. $d \cap \bigcup \operatorname{atoms} a \subseteq c \cap \bigcup \operatorname{atoms} a = 0$. But $d \in \operatorname{atoms} a$. Contradiction. □

Theorem 21 *Any atomistic lattice is atomically separable.*

Proof Let $\mathfrak{A}$ be an atomistic lattice. Let $a, b \in \mathfrak{A}$, $a \subset b$. Then $\bigcup \operatorname{atoms} a \subset \bigcup \operatorname{atoms} b$ and consequently $\operatorname{atoms} a \subset \operatorname{atoms} b$. □

Theorem 22 *Let $\mathfrak{A}$ be an atomic meet-semilattice with least element. Then the following statements are equivalent:*

1. *$\mathfrak{A}$ is separable.*

2. *$\mathfrak{A}$ is atomically separable.*

3. *$\mathfrak{A}$ conforms to Wallman's disjunction property.*

4. *$\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow \exists c \in \mathfrak{A} \setminus \{0\} : (c \asymp a \wedge c \subseteq b))$.*

Proof

(1)⇔(3)⇔(4) Proved above.

(2)⇒(4) Let our semilattice be atomically separable. Let $a \subset b$. Then $\operatorname{atoms} a \subset \operatorname{atoms} b$ and so exists $c \in \operatorname{atoms} b$ such that $c \notin \operatorname{atoms} a$. $c \neq 0$ and $c \subseteq b$; $c \nsubseteq a$, from which (taking in account that c is an atom) $c \subseteq b$ and $c \cap a = 0$. So our semilattice conforms to the formula (4).

(4)⇒(2) Let formula (4) holds. Then for any elements $a \subset b$ exists $c \neq 0$ such that $c \subseteq b$ and $c \cap a = 0$. Because $\mathfrak{A}$ is atomic there exists atom $d \subseteq c$. $d \in \operatorname{atoms} b$ and $d \notin \operatorname{atoms} a$. So $\operatorname{atoms} a \neq \operatorname{atoms} b$ and $\operatorname{atoms} a \subseteq \operatorname{atoms} b$. Consequently $\operatorname{atoms} a \subset \operatorname{atoms} b$.

□

4. Filtrators

Definition 31 *I will call a **filtrator** a pair* $(\mathfrak{A}; \mathfrak{Z})$ *of a poset* $\mathfrak{A}$ *and its subset* $\mathfrak{Z} \subseteq \mathfrak{A}$. *I call* $\mathfrak{A}$ *the **base** of a filtrator and* $\mathfrak{Z}$ *the **core** of a filtrator.*

Definition 32 *I will call a **lattice filtrator** a pair* $(\mathfrak{A}; \mathfrak{Z})$ *of a lattice* $\mathfrak{A}$ *and its subset* $\mathfrak{Z} \subseteq \mathfrak{A}$.

Definition 33 *I will call a **complete lattice filtrator** a pair* $(\mathfrak{A}; \mathfrak{Z})$ *of a complete lattice* $\mathfrak{A}$ *and its subset* $\mathfrak{Z} \subseteq \mathfrak{A}$.

Definition 34 *I will call a **central filtrator** a filtrator* $(\mathfrak{A}; Z(\mathfrak{A}))$ *where* $Z(\mathfrak{A})$ *is the center of a bounded lattice* $\mathfrak{A}$.

Remark 8 One use of filtrators is the theory of filters where the base lattice (or the lattice of principal filters) is essentially considered as the core of the lattice of filters. See below for a more exact formulation. Our primary interest is the properties of filters on sets (that is the filtrator of filters on a set), but instead we will research more general theory of filtrators.

Remark 9 An other important example of filtrators is **filtrator of funcoids** whose base is the set of funcoids [11] and whose core is the set of binary relations (or discrete funcoids).

Definition 35 *I will call **element** of a filtrator an element of its base.*

Definition 36 $\operatorname{up} a = \{c \in \mathfrak{Z} \mid c \supseteq a\}$ *where* $a \in \mathfrak{A}$.

Definition 37 $\operatorname{down} a = \{c \in \mathfrak{Z} \mid c \subseteq a\}$ *where* $a \in \mathfrak{A}$.

Obvious 8 *"*up*" and "*down*" are dual.*

The main purpose of this text is knowing properties of the core of a filtrator to infer properties of the base of the filtrator, specifically properties of $\operatorname{up} a$ for every element a.

Definition 38 *I call a filtrator with **join-closed core** such filtrator* $(\mathfrak{A}; \mathfrak{Z})$ *that* $\bigcup^{\mathfrak{Z}} S = \bigcup^{\mathfrak{A}} S$ *whenever* $\bigcup^{\mathfrak{Z}} S$ *exists for* $S \in \mathscr{P}\mathfrak{Z}$.

Definition 39 *I call a filtrator with **meet-closed core** such filtrator* $(\mathfrak{A};\mathfrak{Z})$ *that* $\bigcap^{\mathfrak{Z}} S = \bigcap^{\mathfrak{A}} S$ *whenever* $\bigcap^{\mathfrak{Z}} S$ *exists for* $S \in \mathcal{P}\mathfrak{Z}$.

Definition 40 *I call a filtrator with **finitely join-closed core** such filtrator* $(\mathfrak{A};\mathfrak{Z})$ *that* $a \cup^{\mathfrak{Z}} b = a \cup^{\mathfrak{A}} b$ *whenever* $a \cup^{\mathfrak{Z}} b$ *exists for* $a, b \in \mathfrak{Z}$.

Definition 41 *I call a filtrator with **finitely meet-closed core** such filtrator* $(\mathfrak{A};\mathfrak{Z})$ *that* $a \cap^{\mathfrak{Z}} b = a \cap^{\mathfrak{A}} b$ *whenever* $a \cap^{\mathfrak{Z}} b$ *exists for* $a, b \in \mathfrak{Z}$.

Definition 42 ***Filtered filtrator*** *is a filtrator* $(\mathfrak{A};\mathfrak{Z})$ *such that* $\forall a \in \mathfrak{A} : a = \bigcap^{\mathfrak{A}} \operatorname{up} a$.

Definition 43 ***Prefiltered filtrator*** *is a filtrator* $(\mathfrak{A};\mathfrak{Z})$ *such that "*up*" is injective.*

Definition 44 ***Semifiltered filtrator*** *is a filtrator* $(\mathfrak{A};\mathfrak{Z})$ *such that*

$$\forall a, b \in \mathfrak{A} : (\operatorname{up} a \supseteq \operatorname{up} b \Rightarrow a \subseteq b).$$

Obvious 9

- *Every filtered filtrator is semifiltered.*
- *Every semifiltered filtrator is prefiltered.*

Obvious 10 *"*up*" is a straight map from* $\mathfrak{A}$ *to the dual of the poset* $\mathcal{P}\mathfrak{Z}$ *if* $(\mathfrak{A};\mathfrak{Z})$ *is a semifiltered filtrator.*

Theorem 23 *Each semifiltered filtrator is a filtrator with join-closed core.*

Proof Let $(\mathfrak{A};\mathfrak{Z})$ be a semifiltered filtrator. Let $S \in \mathcal{P}\mathfrak{Z}$ and $\bigcup^{\mathfrak{Z}} S$ is defined. We need to prove $\bigcup^{\mathfrak{A}} S = \bigcup^{\mathfrak{Z}} S$. That $\bigcup^{\mathfrak{Z}} S$ is an upper bound for S is obvious. Let $a \in \mathfrak{A}$ be an upper bound for S. Enough to prove that $\bigcup^{\mathfrak{Z}} S \subseteq a$. Really,

$$c \in \operatorname{up} a \Rightarrow c \supseteq a \Rightarrow \forall x \in S : c \supseteq x \Rightarrow c \supseteq \bigcup{}^{\mathfrak{Z}} S \Rightarrow c \in \operatorname{up} \bigcup{}^{\mathfrak{Z}} S;$$

so $\operatorname{up} a \subseteq \operatorname{up} \bigcup^{\mathfrak{Z}} S$ and thus $a \supseteq \bigcup^{\mathfrak{Z}} S$ because it is semifiltered. □

4.1. Core part

Definition 45 *The **core part** of an element* $a \in \mathfrak{A}$ *is* $\operatorname{Cor} a = \bigcap^{\mathfrak{Z}} \operatorname{up} a$.

Definition 46 *The **dual core part** of an element* $a \in \mathfrak{A}$ *is* $\operatorname{Cor}' a = \bigcup^{\mathfrak{Z}} \operatorname{down} a$.

Obvious 11 Cor' *is dual of* Cor.

Theorem 24 $\operatorname{Cor} a \subseteq a$ *whenever* $\operatorname{Cor} a$ *exists for any element* a *of a filtered filtrator.*

Proof $\operatorname{Cor} a = \bigcap^{\mathfrak{Z}} \operatorname{up} a \subseteq \bigcap^{\mathfrak{A}} \operatorname{up} a = a$. □

Corollary 6 *$\operatorname{Cor} a \in \operatorname{down} a$ whenever $\operatorname{Cor} a$ exists for any element a of a filtered filtrator.*

Theorem 25 *$\operatorname{Cor}' a \subseteq a$ whenever $\operatorname{Cor}' a$ exists for any element a of a filtrator with join-closed core.*

Proof $\operatorname{Cor}' a = \bigcup^{\mathfrak{Z}} \operatorname{down} a = \bigcup^{\mathfrak{A}} \operatorname{down} a \subseteq a$. □

Corollary 7 *$\operatorname{Cor}' a \in \operatorname{down} a$ whenever $\operatorname{Cor}' a$ exists for any element a of a filtrator with join-closed core.*

Proposition 15 *$\operatorname{Cor}' a \subseteq \operatorname{Cor} a$ whenever both $\operatorname{Cor} a$ and $\operatorname{Cor}' a$ exist for any element a of a filtrator with join-closed core.*

Proof $\operatorname{Cor} a = \bigcap^{\mathfrak{Z}} \operatorname{up} a \supseteq \operatorname{Cor}' a$ because $\forall A \in \operatorname{up} a : \operatorname{Cor}' a \subseteq A$. □

Theorem 26 *$\operatorname{Cor}' a = \operatorname{Cor} a$ whenever both $\operatorname{Cor} a$ and $\operatorname{Cor}' a$ exist for any element a of a filtered filtrator.*

Proof It is with join-closed core because it is semifiltered. So $\operatorname{Cor}' a \subseteq \operatorname{Cor} a$. $\operatorname{Cor} a \in \operatorname{down} a$. So $\operatorname{Cor} a \subseteq \bigcup^{\mathfrak{Z}} \operatorname{down} a = \operatorname{Cor}' a$. □

Obvious 12 *$\operatorname{Cor}' a = \max \operatorname{down} a$ for an element a of a filtrator with join-closed core.*

4.2. Filtrators with separable core

Definition 47 *Let $\mathfrak{A}$ be a filtrator. $\mathfrak{A}$ is a **filtrator with separable core** when*

$$\forall x, y \in \mathfrak{A} : (x \asymp^{\mathfrak{A}} y \Rightarrow \exists X \in \operatorname{up} x : X \asymp^{\mathfrak{A}} y).$$

Proposition 16 *Let $\mathfrak{A}$ be a filtrator. $\mathfrak{A}$ is a filtrator with separable core iff*

$$\forall x, y \in \mathfrak{A} : (x \asymp^{\mathfrak{A}} y \Rightarrow \exists X \in \operatorname{up} x, Y \in \operatorname{up} y : X \asymp^{\mathfrak{A}} Y).$$

Proof

$\Rightarrow$ Apply the definition twice.

$\Leftarrow$ Obvious.

□

Definition 48 *Let $\mathfrak{A}$ be a filtrator. $\mathfrak{A}$ is a **filtrator with co-separable core** when*

$$\forall x, y \in \mathfrak{A} : (x \equiv^{\mathfrak{A}} y \Rightarrow \exists X \in \operatorname{down} x : X \equiv^{\mathfrak{A}} y).$$

Obvious 13 *Co-separability is the dual of separability.*

Proposition 17 *Let $\mathfrak{A}$ be a filtrator. $\mathfrak{A}$ is a filtrator with co-separable core iff*

$$\forall x, y \in \mathfrak{A} : (x \equiv^{\mathfrak{A}} y \Rightarrow \exists X \in \operatorname{down} x, Y \in \operatorname{down} y : X \equiv^{\mathfrak{A}} Y).$$

Proof By duality. □

4.3. Intersecting and joining with an element of the core

Definition 49 *I call* ***down-aligned*** *filtrator such a filtrator $(\mathfrak{A}; \mathfrak{Z})$ that $\mathfrak{A}$ and $\mathfrak{Z}$ have common least element. (Let's denote it 0.)*

Definition 50 *I call* ***up-aligned*** *filtrator such a filtrator $(\mathfrak{A}; \mathfrak{Z})$ that $\mathfrak{A}$ and $\mathfrak{Z}$ have common greatest element. (Let's denote it 1.)*

Theorem 27 *For a filtrator $(\mathfrak{A}; \mathfrak{Z})$ where $\mathfrak{Z}$ is a boolean lattice, for every $B \in \mathfrak{Z}$, $\mathcal{A} \in \mathfrak{A}$:*

1. *$B \asymp^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \supseteq \mathcal{A}$ if it is down-aligned, with finitely meet-closed and separable core;*
2. *$B \equiv^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \subseteq \mathcal{A}$ if it is up-aligned, with finitely join-closed and co-separable core.*

Proof We will prove only the first as the second is dual.

$$\begin{aligned} B \asymp^{\mathfrak{A}} \mathcal{A} &\Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A} : B \asymp^{\mathfrak{A}} A &\Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A} : B \cap^{\mathfrak{A}} A = 0 &\Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A} : B \cap^{\mathfrak{Z}} A = 0 &\Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A} : \overline{B} \supseteq A &\Leftrightarrow \\ \overline{B} \in \operatorname{up} \mathcal{A} &\Leftrightarrow \\ \overline{B} \supseteq \mathcal{A}. \end{aligned}$$

□

5. Filters

5.1. Filters on posets

Let $\mathfrak{A}$ be a poset (partially ordered set) with the partial order $\subseteq$. I will call it **the base poset**.

Definition 51 ***Filter base*** *is a nonempty subset F of $\mathfrak{A}$ such that*

$$\forall X, Y \in F \exists Z \in F : (Z \subseteq X \wedge Z \subseteq Y).$$

Obvious 14 *A nonempty chain is a filter base.*

Definition 52 ***Upper set*** *is a subset F of $\mathfrak{A}$ such that*

$$\forall X \in F, Y \in \mathfrak{A} : (Y \supseteq X \Rightarrow Y \in F).$$

Definition 53 *Filter is a subset of $\mathfrak{A}$ which is both filter base and upper set. I will denote the set of filters $\mathfrak{f}$.*

Proposition 18 *If 1 is the maximal element of $\mathfrak{A}$ then $1 \in F$ for any filter F.*

Proof If $1 \notin F$ then $\forall K \in \mathfrak{A} : K \notin F$ and so F is empty what is impossible. □

Proposition 19 *Let S be a filter base. If $A_0, \ldots, A_n \in S$ $(n \in \mathbb{N})$, then*

$$\exists C \in S : (C \subseteq A_0 \wedge ... \wedge C \subseteq A_n).$$

Proof It can be easily proved by induction. □

The dual of filters is called **ideals**. We do not use ideals in this work however.

5.2. Filters on meet-semilattice

Theorem 28 *If $\mathfrak{A}$ is a meet-semilattice and F is a nonempty subset of $\mathfrak{A}$ then the following conditions are equivalent:*

1. *F is a filter.*

2. *$\forall X, Y \in F : X \cap Y \in F$ and F is an upper set.*

3. *$\forall X, Y \in \mathfrak{A} : (X, Y \in F \Leftrightarrow X \cap Y \in F)$.*

Proof

(1)⇒(2) Let F be a filter. Then F is an upper set. If $X, Y \in F$ then $Z \subseteq X \wedge Z \subseteq Y$ for some $Z \in F$. Because F is an upper set and $Z \subseteq X \cap Y$ then $X \cap Y \in F$.

(2)⇒(1) Let $\forall X, Y \in F : X \cap Y \in F$ and F is an upper set. We need to prove that F is a filter base. But it is obvious taking $Z = X \cap Y$ (we have also taken in account that $F \neq \emptyset$).

(2)⇒(3) Let $\forall X, Y \in F : X \cap Y \in F$ and F is an upper set. Then

$$\forall X, Y \in \mathfrak{A} : (X, Y \in F \Rightarrow X \cap Y \in F).$$

Let $X \cap Y \in F$; then $X, Y \in F$ because F is an upper set.

(3)⇒(2) Let
$$\forall X, Y \in \mathfrak{A} : (X, Y \in F \Leftrightarrow X \cap Y \in F).$$

Then $\forall X, Y \in F : X \cap Y \in F$. Let $X \in F$ and $X \subseteq Y \in \mathfrak{A}$. Then $X \cap Y = X \in F$. Consequently $X, Y \in F$. So F is an upper set.

□

Proposition 20 *Let $\mathfrak{A}$ be a meet-semilattice. Let S be a filter base. If $A_0, \ldots, A_n \in S$ $(n \in \mathbb{N})$, then*
$$\exists C \in S : C \subseteq A_0 \cap \ldots \cap A_n.$$

Proof It can be easily proved by induction. □

Proposition 21 *If $\mathfrak{A}$ is a meet-semilattice and S is a filter base, $A \in \mathfrak{A}$, then $\langle A\cap \rangle S$ is also a filter base.*

Proof $\langle A\cap \rangle S \neq \emptyset$ because $S \neq \emptyset$.

Let $X, Y \in \langle A\cap \rangle S$. Then $X = A \cap X'$ and $Y = A \cap Y'$ where $X', Y' \in S$. Exists $Z' \in S$ such that $Z' \subseteq X' \cap Y'$. So $X \cap Y = A \cap X' \cap Y' \supseteq A \cap Z' \in \langle A\cap \rangle S$.
□

5.3. Characterization of finitely meet-closed filtrators

Theorem 29 *The following are equivalent for a filtrator $(\mathfrak{A}; \mathfrak{Z})$ whose core is a meet-semilattice such that $\forall a \in \mathfrak{A} : \operatorname{up} a \neq \emptyset$:*

1. *The filtrator is finitely meet-closed.*

2. $\operatorname{up} a$ *is a filter on* $\mathfrak{Z}$ *for every* $a \in \mathfrak{A}$.

Proof

(1)⇒(2) Let $X, Y \in \operatorname{up} a$. Then $X \cap^{\mathfrak{Z}} Y = X \cap^{\mathfrak{A}} Y \supseteq a$. That $\operatorname{up} a$ is an upper set is obvious. So taking in account that $\operatorname{up} a \neq \emptyset$, $\operatorname{up} a$ is a filter.

(2)⇒(1) It is enough to prove that $a \subseteq A, B \Rightarrow a \subseteq A \cap^{\mathfrak{Z}} B$ for every $A, B \in \mathfrak{A}$. Really:
$$a \subseteq A, B \Rightarrow A, B \in \operatorname{up} a \Rightarrow A \cap^{\mathfrak{Z}} B \in \operatorname{up} a \Rightarrow a \subseteq A \cap^{\mathfrak{Z}} B.$$

□

6. Filter objects

I want to equate principal filters (see below) with the elements of the base poset. Such thing can be done using the principles described in the appendix Appendix B. The formal definitions follow.

6.1. Definition of filter objects

Let $\mathfrak{A}$ be a poset.

Definition 54 *Let* $\uparrow a \stackrel{\mathrm{def}}{=} \{x \in \mathfrak{A} \mid x \supseteq a\}$ *for every* $a \in \mathfrak{A}$*. Elements of the set* $\langle\uparrow\rangle \mathfrak{A}$ *are called* ***principal filters****.*

Obvious 15 $\uparrow$ *is an injection from* $\mathfrak{A}$ *to* $\mathfrak{f}$*.*

Let M be a bijection defined on $\mathfrak{f}$ such that $M \circ \uparrow = \mathrm{id}_{\mathfrak{A}}$. (See the appendix Appendix B for a proof that such a bijection exists.)

Definition 55 *Let* $\mathfrak{F} = \operatorname{im} M$*. I call elements of* $\mathfrak{F}$ *as* ***filter objects*** *(f.o. for short).*

Remark 10 Below we will show that $\operatorname{up} \mathcal{A} = M^{-1}\mathcal{A}$ for each $\mathcal{A} \in \mathfrak{F}$.

Obvious 16 $\uparrow = M^{-1}|_{\mathfrak{A}}$*.*

Obvious 17 M^{-1} *is a bijection* $\mathfrak{F} \to \mathfrak{f}$*.*

Proposition 22 $\mathfrak{A} \subseteq \mathfrak{F}$*.*

Proof $x \in \mathfrak{A} \Rightarrow M \uparrow x = x \Rightarrow x \in \operatorname{im} M \Rightarrow x \in \mathfrak{F}$. □

6.2. Order of filter objects

Proposition 23 $a \subseteq b \Leftrightarrow M^{-1}a \supseteq M^{-1}b$*.*

Proof $a \subseteq b \Leftrightarrow \uparrow a \supseteq \uparrow b \Leftrightarrow M^{-1}a \supseteq M^{-1}b$. □

As a generalization of the last proposition we may define the order on $\mathfrak{F}$:

Definition 56 $\mathcal{A} \subseteq \mathcal{B} \stackrel{\mathrm{def}}{=} M^{-1}\mathcal{A} \supseteq M^{-1}\mathcal{B}$ *for all* $\mathcal{A}, \mathcal{B} \in \mathfrak{A}$*.*

I will call the pair $(\mathfrak{F}; \mathfrak{A})$ the **primary filtrator**.

Theorem 30 *For the primary filtrator* $(\mathfrak{F}; \mathfrak{A})$ *we have* $\operatorname{up} \mathcal{A} = M^{-1}\mathcal{A}$ *for each* $\mathcal{A} \in \mathfrak{F}$*.*

Proof $x \in \operatorname{up} \mathcal{A} \Leftrightarrow x \supseteq \mathcal{A} \Leftrightarrow M^{-1}x \subseteq M^{-1}\mathcal{A} \Leftrightarrow \uparrow x \subseteq M^{-1}\mathcal{A} \Leftrightarrow x \in M^{-1}\mathcal{A}$ for every $x \in \mathfrak{A}$. □

So we have:

- "up" is a bijection from $\mathfrak{F}$ to $\mathfrak{f}$.
- $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \operatorname{up} \mathcal{A} \supseteq \operatorname{up} \mathcal{B}$ for each $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.
- $\operatorname{up} a = \uparrow a$ for every $a \in \mathfrak{A}$.

A filter object $\mathcal{A}$ is represented by the value of $\operatorname{up}\mathcal{A}$. We are not interested in the internal structure of filter objects (which can be inferred from the appendix Appendix B), but only in the value of $\operatorname{up}\mathcal{A}$. Thus the name "filter objects" by analogy with an object in object oriented programming where an object is completely characterized by its methods, likewise a filter object $\mathcal{A}$ is completely characterized by $\operatorname{up}\mathcal{A}$.

7. Lattice of filter objects

7.1. Minimal and maximal f.o.

Obvious 18 *The filter object* $0 = \operatorname{up}^{-1}\mathfrak{A}$ *(equal to the least element of the poset* $\mathfrak{A}$ *if this least exists) is the least element of the poset of filter objects.*

Proposition 24 *If there exists greatest element* 1 *of the poset* $\mathfrak{A}$ *then it is also the greatest element of the poset of filter objects.*

Proof Take in account that filters are nonempty. □

Obvious 19 *1. If the base poset has least element, then the primary filtrator is down-aligned.*

2. If the base poset has greatest element, then the primary filtrator is up-aligned.

7.2. Primary filtrator is filtered

Theorem 31 *Every primary filtrator is filtered.*

Proof We need to prove that $\mathcal{A} = \bigcap^{\mathfrak{F}} \operatorname{up}\mathcal{A}$ for every $\mathcal{A} \in \mathfrak{F}$.

$\mathcal{A}$ is obviously a lower bound for $\operatorname{up}\mathcal{A}$.

Let $\mathcal{B}$ be a lower bound for $\operatorname{up}\mathcal{A}$ that is $\forall K \in \operatorname{up}\mathcal{A} : K \supseteq \mathcal{B}$. Then $\operatorname{up}\mathcal{A} \subseteq \operatorname{up}\mathcal{B}$; $\mathcal{A} \supseteq \mathcal{B}$. So $\mathcal{A}$ is the greatest lower bound of $\operatorname{up}\mathcal{A}$. □

7.3. Formulas for meets and joins of filter objects

Lemma 2 *If* f *is an order embedding from a poset* $\mathfrak{A}$ *to a complete lattice* $\mathfrak{B}$ *and* $S \in \mathcal{P}\mathfrak{A}$ *and exists such* $\mathcal{F} \in \mathfrak{A}$ *that* $f\mathcal{F} = \bigcup^{\mathfrak{B}} \langle f \rangle S$*, then* $\bigcup^{\mathfrak{A}} S$ *exists and* $f \bigcup^{\mathfrak{A}} S = \bigcup^{\mathfrak{B}} \langle f \rangle S$.

Proof f is an order isomorphism from $\mathfrak{A}$ to $\mathfrak{B}|_{\langle f \rangle \mathfrak{A}}$. $f\mathcal{F} \in \mathfrak{B}|_{\langle f \rangle \mathfrak{A}}$.

Consequently, $\bigcup^{\mathfrak{B}} \langle f \rangle S \in \mathfrak{B}|_{\langle f \rangle \mathfrak{A}}$ and $\bigcup^{\mathfrak{B}|_{\langle f \rangle \mathfrak{A}}} \langle f \rangle S = \bigcup^{\mathfrak{B}} \langle f \rangle S$.

$f \bigcup^{\mathfrak{A}} S = \bigcup^{\mathfrak{B}|_{\langle f \rangle \mathfrak{A}}} \langle f \rangle S$ because f is an order isomorphism.

Combining, $f \bigcup^{\mathfrak{A}} S = \bigcup^{\mathfrak{B}} \langle f \rangle S$. □

Theorem 32 *If* $\mathfrak{A}$ *is a meet-semilattice with greatest element* 1 *then* $\bigcup^{\mathfrak{F}} S$ *exists and* $\operatorname{up} \bigcup^{\mathfrak{F}} S = \bigcap^{\mathcal{P}\mathfrak{A}} \langle \operatorname{up} \rangle S$ *for every* $S \in \mathcal{P}\mathfrak{F}$.

Proof Taking in account the lemma it is enough to prove that exists $\mathcal{F} \in \mathfrak{F}$ such that $\operatorname{up}\mathcal{F} = \bigcap^{\mathcal{P}\mathfrak{A}} \langle \operatorname{up} \rangle S$, that is that $R = \bigcap^{\mathcal{P}\mathfrak{A}} \langle \operatorname{up} \rangle S$ is a filter.

R is nonempty because $1 \in R$. Let $A, B \in R$; then $\forall \mathcal{F} \in S : A, B \in \operatorname{up}\mathcal{F}$, consequently $\forall \mathcal{F} \in S : A \cap^{\mathfrak{A}} B \in \operatorname{up}\mathcal{F}$. Consequently $A \cap^{\mathfrak{A}} B \in \bigcap^{\mathcal{P}\mathfrak{A}} \langle \operatorname{up} \rangle S = R$. So R is a filter base. Let $X \in R$ and $X \subseteq Y \in \mathfrak{A}$; then $\forall \mathcal{F} \in S : X \in \operatorname{up}\mathcal{F}$; $\forall \mathcal{F} \in S : Y \in \operatorname{up}\mathcal{F}$; $Y \in R$. So R is an upper set. □

Corollary 8 *If $\mathfrak{A}$ is a meet-semilattice with greatest element 1 then $\mathfrak{F}$ is a complete lattice.*

Corollary 9 *If $\mathfrak{A}$ is a meet-semilattice with greatest element 1 then for any $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$*

$$\operatorname{up}(\mathcal{A} \cup^{\mathfrak{F}} \mathcal{B}) = \operatorname{up}\mathcal{A} \cap \operatorname{up}\mathcal{B}.$$

Theorem 33 *If $\mathfrak{A}$ is a join-semilattice then $\mathfrak{F}$ is a join-semilattice and for any $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$*

$$\operatorname{up}(\mathcal{A} \cup^{\mathfrak{F}} \mathcal{B}) = \operatorname{up}\mathcal{A} \cap \operatorname{up}\mathcal{B}.$$

Proof Taking in account the lemma it is enough to prove that $R = \operatorname{up}\mathcal{A} \cap \operatorname{up}\mathcal{B}$ is a filter.

R is nonempty because exist $X \in \operatorname{up}\mathcal{A}$ and $Y \in \operatorname{up}\mathcal{B}$ and $R \ni X \cup^{\mathfrak{A}} Y$.

Let $A, B \in R$. Then $A, B \in \operatorname{up}\mathcal{A}$; so exists $C \in \operatorname{up}\mathcal{A}$ such that $C \subseteq A \wedge C \subseteq B$. Analogously exists $D \in \operatorname{up}\mathcal{B}$ such that $D \subseteq A \wedge D \subseteq B$. Let $E = C \cup^{\mathfrak{A}} D$. Then $E \in \operatorname{up}\mathcal{A}$ and $E \in \operatorname{up}\mathcal{B}$; $E \in R$ and $E \subseteq A \wedge E \subseteq B$. So R is a filter base.

That R is an upper set is obvious. □

Theorem 34 *If $\mathfrak{A}$ is a distributive lattice then for $S \in \mathcal{P}\mathfrak{F} \setminus \{\emptyset\}$*

$$\operatorname{up} \bigcap{}^{\mathfrak{F}} S = \left\{ K_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} K_n \;\middle|\; K_i \in \bigcup \langle \operatorname{up} \rangle S \textit{ where } i = 0, \ldots, n \textit{ for } n \in \mathbb{N} \right\}.$$

Proof Let's denote the right part of the equality to be proven as R. First we will prove that R is a filter. R is nonempty because S is nonempty.

Let $A, B \in R$. Then $A = X_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} X_k$, $B = Y_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} Y_l$ where $X_i, Y_j \in \bigcup \langle \operatorname{up} \rangle S$. So

$$A \cap^{\mathfrak{A}} B = X_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} X_k \cap^{\mathfrak{A}} Y_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} Y_l \in R.$$

Let $R \ni C \supseteq A$. Consequently (distributivity used)

$$C = C \cup^{\mathfrak{A}} A = (C \cup^{\mathfrak{A}} X_0) \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} (C \cup^{\mathfrak{A}} X_k).$$

$X_i \in \operatorname{up} P$ for some $P \in S$; $C \cup^{\mathfrak{A}} X_i \in \operatorname{up} P$; consequently $C \in \operatorname{up} P$; $C \in R$.

We have proved that R is a filter base and an upper set. So R is a filter.

Consequently the statement of our theorem is equivalent to $\bigcap^{\mathfrak{F}} S = \operatorname{up}^{-1} R$.

Let $\mathcal{A} \in S$. Then $\operatorname{up}\mathcal{A} \in \langle \operatorname{up} \rangle S$; $\operatorname{up}\mathcal{A} \subseteq \bigcup \langle \operatorname{up} \rangle S$;

$$R \supseteq \left\{ K_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} K_n \;\middle|\; K_i \in \operatorname{up}\mathcal{A} \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N} \right\} = \operatorname{up}\mathcal{A}.$$

Consequently $\mathcal{A} \supseteq \operatorname{up}^{-1} R$.

Let now $\mathcal{B} \in \mathfrak{F}$ and $\forall \mathcal{A} \in S : \mathcal{A} \supseteq \mathcal{B}$. Then $\forall \mathcal{A} \in S : \operatorname{up} \mathcal{B} \supseteq \operatorname{up} \mathcal{A}$. $\operatorname{up} \mathcal{B} \supseteq \bigcup \langle \operatorname{up} \rangle S$. From this $\operatorname{up} \mathcal{B} \supseteq T$ for any finite set $T \subseteq \bigcup \langle \operatorname{up} \rangle S$. Consequently $\operatorname{up} \mathcal{B} \ni \bigcap^{\mathfrak{A}} T$. Thus $\operatorname{up} \mathcal{B} \supseteq R$; $\mathcal{B} \subseteq \operatorname{up}^{-1} R$.

Comparing we get $\bigcap^{\mathfrak{F}} S = \operatorname{up}^{-1} R$. □

Theorem 35 *If $\mathfrak{A}$ is a distributive lattice then for any $\mathcal{F}_0, \ldots, \mathcal{F}_m \in \mathfrak{F}$ $(m \in \mathbb{N})$*

$$\operatorname{up}(\mathcal{F}_0 \cap^{\mathfrak{F}} \ldots \cap^{\mathfrak{F}} \mathcal{F}_m) = \left\{ K_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} K_m \mid K_i \in \operatorname{up} \mathcal{F}_i, i = 0, \ldots, m \right\}.$$

Proof Let's denote the right part of the equality to be proven as R. First we will prove that R is a filter. Obviously R is nonempty.

Let $A, B \in R$. Then $A = X_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} X_m$, $B = Y_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} Y_m$ where $X_i, Y_i \in \operatorname{up} \mathcal{F}_i$.

$$A \cap^{\mathfrak{A}} B = (X_0 \cap^{\mathfrak{A}} Y_0) \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} (X_m \cap^{\mathfrak{A}} Y_m),$$

consequently $A \cap^{\mathfrak{A}} B \in R$.

Let $R \ni C \supseteq A$.

$$C = A \cup^{\mathfrak{A}} C = (X_0 \cup^{\mathfrak{A}} C) \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} (X_m \cup^{\mathfrak{A}} C) \in R.$$

So R is a filter. Consequently the statement of our theorem is equivalent to

$$\mathcal{F}_0 \cap^{\mathfrak{F}} \ldots \cap^{\mathfrak{F}} \mathcal{F}_m = \operatorname{up}^{-1} R.$$

Let $P_i \in \operatorname{up} \mathcal{F}_i$. Then $P_i \in R$ because $P_i = (P_i \cup^{\mathfrak{A}} P_0) \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} (P_i \cup^{\mathfrak{A}} P_m)$. So $\operatorname{up} \mathcal{F}_i \subseteq R$; $\mathcal{F}_i \supseteq \operatorname{up}^{-1} R$.

Let now $\mathcal{B} \in \mathfrak{F}$ and $\forall i \in \{0, \ldots, m\} : \mathcal{F}_i \supseteq \mathcal{B}$. Then $\forall i \in \{0, \ldots, m\} :$ $\operatorname{up} \mathcal{F}_i \subseteq \operatorname{up} \mathcal{B}$.

$L_i \in \operatorname{up} \mathcal{B}$ for any $L_i \in \operatorname{up} \mathcal{F}_i$. $L_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} L_m \in \operatorname{up} \mathcal{B}$. So $\operatorname{up} \mathcal{B} \supseteq R$; $\mathcal{B} \subseteq \operatorname{up}^{-1} R$.

So $\mathcal{F}_0 \cap^{\mathfrak{F}} \ldots \cap^{\mathfrak{F}} \mathcal{F}_m = \operatorname{up}^{-1} R$. □

Definition 57 *I will call a* ***lattice of filter objects on a set*** *a set of filter objects on the lattice of all subsets of a set. (From the above it follows that it is actually a complete lattice.)*

7.4. Distributivity of the lattice of filter objects

Theorem 36 *If $\mathfrak{A}$ is a distributive lattice with greatest element, $S \in \mathscr{P}\mathfrak{F}$ and $\mathcal{A} \in \mathfrak{F}$ then $\mathcal{A} \cup^{\mathfrak{F}} \bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \left\langle \mathcal{A} \cup^{\mathfrak{F}} \right\rangle S$.*

Proof Taking in account the previous subsection, we have:

$$\begin{aligned}
\operatorname{up}\left(\mathcal{A} \cup^{\mathfrak{F}} \bigcap{}^{\mathfrak{F}} S\right) &= \\
\operatorname{up}\mathcal{A} \cap \operatorname{up}\bigcap{}^{\mathfrak{F}} S &= \\
\operatorname{up}\mathcal{A} \cap \left\{K_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} K_n \mid K_i \in \bigcup \langle \operatorname{up} \rangle S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\right\} &= \\
\left\{K_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} K_n \mid K_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} K_n \in \operatorname{up}\mathcal{A}, K_i \in \bigcup \langle \operatorname{up} \rangle S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\right\} &= \\
\left\{K_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} K_n \mid K_i \in \operatorname{up}\mathcal{A}, K_i \in \bigcup \langle \operatorname{up} \rangle S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\right\} &= \\
\left\{K_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} K_n \mid K_i \in \operatorname{up}\mathcal{A}, K_i \in \bigcup \{\operatorname{up}\mathcal{X} \mid \mathcal{X} \in S\} \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\right\} &= \\
\left\{K_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} K_n \mid K_i \in \operatorname{up}\mathcal{A} \cap \bigcup \{\operatorname{up}\mathcal{X} \mid \mathcal{X} \in S\} \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\right\} &= \\
\left\{K_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} K_n \mid K_i \in \bigcup \{\operatorname{up}\mathcal{A} \cap \operatorname{up}\mathcal{X} \mid \mathcal{X} \in S\} \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\right\} &= \\
\left\{K_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} K_n \mid K_i \in \bigcup \{\operatorname{up}(\mathcal{A} \cup^{\mathfrak{F}} \mathcal{X}) \mid \mathcal{X} \in S\} \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\right\} &= \\
\left\{K_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} K_n \mid K_i \in \bigcup \langle \operatorname{up} \rangle \{\mathcal{A} \cup^{\mathfrak{F}} \mathcal{X} \mid \mathcal{X} \in S\} \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\right\} &= \\
\operatorname{up}\bigcap{}^{\mathfrak{F}} \{\mathcal{A} \cup^{\mathfrak{F}} \mathcal{X} \mid \mathcal{X} \in S\}.
\end{aligned}$$

□

Corollary 10 *If $\mathfrak{A}$ is a distributive lattice with greatest element, then $\mathfrak{F}$ is also a distributive lattice.*

Corollary 11 *If $\mathfrak{A}$ is a distributive lattice with greatest element, then $\mathfrak{F}$ is a co-brouwerian lattice.*

7.5. Separability of core for primary filtrators

Theorem 37 *A primary filtrator with least element, whose base is a distributive lattice, is with separable core.*

Proof Let $\mathcal{A} \asymp^{\mathfrak{F}} \mathcal{B}$ where $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

$$\operatorname{up}(\mathcal{A} \cap^{\mathfrak{F}} \mathcal{B}) = \left\{A \cap^{\mathfrak{A}} B \mid A \in \operatorname{up}\mathcal{A}, B \in \operatorname{up}\mathcal{B}\right\}.$$

So

$$\begin{aligned}
\mathcal{A} \asymp^{\mathfrak{F}} \mathcal{B} &\Leftrightarrow \\
0 \in \operatorname{up}(\mathcal{A} \cap^{\mathfrak{F}} \mathcal{B}) &\Leftrightarrow \\
\exists A \in \operatorname{up}\mathcal{A}, B \in \operatorname{up}\mathcal{B} : A \cap^{\mathfrak{A}} B = 0 &\Leftrightarrow \\
\exists A \in \operatorname{up}\mathcal{A}, B \in \operatorname{up}\mathcal{B} : A \cap^{\mathfrak{F}} B = 0 &
\end{aligned}$$

(used the theorem 23). □

Theorem 38 *Let $(\mathfrak{A};\mathfrak{Z})$ be an up-aligned filtered filtrator whose core is a meet infinite distributive complete lattice. Then this filtrator is with co-separable core.*

Proof Our filtrator is with join-closed core.

Let $a,b\in\mathfrak{A}$. $\operatorname{Cor} a$ and $\operatorname{Cor} b$ exist since $\mathfrak{Z}$ is a complete lattice.

$\operatorname{Cor} a\in\operatorname{down} a$ and $\operatorname{Cor} b\in\operatorname{down} b$ by the corollary 6 since our filtrator is filtered. So we have

$$\begin{array}{rcl}
\exists x\in\operatorname{down} a, y\in\operatorname{down} b: x\cup^{\mathfrak{A}} y=1 & \Leftarrow & \\
\operatorname{Cor} a\cup^{\mathfrak{A}}\operatorname{Cor} b=1 & \Leftrightarrow & \text{(by finite join-closedness of the core)}\\
\operatorname{Cor} a\cup^{\mathfrak{Z}}\operatorname{Cor} b=1 & \Leftrightarrow & \\
\bigcap^{\mathfrak{Z}}\operatorname{up} a\cup^{\mathfrak{Z}}\bigcap^{\mathfrak{Z}}\operatorname{up} b=1 & \Leftrightarrow & \text{(by infinite distributivity)}\\
\bigcap^{\mathfrak{Z}}\left\{x\cup^{\mathfrak{Z}} y\mid x\in\operatorname{up} a, y\in\operatorname{up} b\right\}=1 & \Leftrightarrow & \\
\forall x\in\operatorname{up} a, y\in\operatorname{up} b: x\cup^{\mathfrak{Z}} y=1 & \Leftrightarrow & \text{(by finite join-closedness of the core)}\\
\forall x\in\operatorname{up} a, y\in\operatorname{up} b: x\cup^{\mathfrak{A}} y=1 & \Leftarrow & \\
a\cup^{\mathfrak{A}} b=1. & &
\end{array}$$

□

7.6. Filters over boolean lattices

Theorem 39 *If $\mathfrak{A}$ is a boolean lattice then $a\setminus^{\mathfrak{F}} B=a\cap^{\mathfrak{F}}\overline{B}$ (where the complement is taken on $\mathfrak{A}$).*

Proof $\mathfrak{F}$ is distributive by the theorem 10. Our filtrator is with finitely meet-closed core by the theorem 29 and with join-closed core by the theorem 23.

$(a\cap^{\mathfrak{F}}\overline{B})\cup^{\mathfrak{F}} B=(a\cup^{\mathfrak{F}} B)\cap^{\mathfrak{F}}(\overline{B}\cup^{\mathfrak{F}} B)=(a\cup^{\mathfrak{F}} B)\cap^{\mathfrak{F}}(\overline{B}\cup^{\mathfrak{A}} B)=(a\cup^{\mathfrak{F}} B)\cap^{\mathfrak{F}} 1=a\cup^{\mathfrak{F}} B$.

$(a\cap^{\mathfrak{F}}\overline{B})\cap^{\mathfrak{F}} B=a\cap^{\mathfrak{F}}(\overline{B}\cap^{\mathfrak{F}} B)=a\cap^{\mathfrak{F}}(\overline{B}\cap^{\mathfrak{A}} B)=a\cap^{\mathfrak{F}} 0=0$.

So $a\cap^{\mathfrak{F}}\overline{B}$ is the difference of a and B. □

7.7. Distributivity for an element of boolean core

Lemma 3 *Let $\mathfrak{F}$ be the set of filter objects over a boolean lattice $\mathfrak{A}$.*

Then $A\cap^{\mathfrak{F}}$ is a lower adjoint of $\overline{A}\cup^{\mathfrak{F}}$ for every $A\in\mathfrak{A}$.

Proof We will use the theorem 8.

That $A\cap^{\mathfrak{F}}$ and $\overline{A}\cup^{\mathfrak{F}}$ are monotone is obvious.

We need to prove (for every $x,y\in\mathfrak{F}$) that

$$x\subseteq\overline{A}\cup^{\mathfrak{F}}(A\cap^{\mathfrak{F}} x)\quad\text{and}\quad A\cap^{\mathfrak{F}}(\overline{A}\cup^{\mathfrak{F}} y)\subseteq y.$$

Really, $\overline{A}\cup^{\mathfrak{F}}(A\cap^{\mathfrak{F}} x)=(\overline{A}\cup^{\mathfrak{F}} A)\cap^{\mathfrak{F}}(\overline{A}\cup^{\mathfrak{F}} x)=(\overline{A}\cup^{\mathfrak{A}} A)\cap^{\mathfrak{F}}(\overline{A}\cup^{\mathfrak{F}} x)=1\cap^{\mathfrak{F}}(\overline{A}\cup^{\mathfrak{F}} x)=\overline{A}\cup^{\mathfrak{F}} x\supseteq x$ and $A\cap^{\mathfrak{F}}(\overline{A}\cup^{\mathfrak{F}} y)=(A\cap^{\mathfrak{F}}\overline{A})\cup^{\mathfrak{F}}(A\cap^{\mathfrak{F}} y)=(A\cap^{\mathfrak{A}}\overline{A})\cup^{\mathfrak{F}}(A\cap^{\mathfrak{F}} y)=0\cup^{\mathfrak{F}}(A\cap^{\mathfrak{F}} y)=A\cap^{\mathfrak{F}} y\subseteq y$. □

Theorem 40 *Let $\mathfrak{F}$ be the set of filter objects over a boolean lattice $\mathfrak{A}$.*
$A \cap^{\mathfrak{F}} \bigcup^{\mathfrak{F}} S = \bigcup^{\mathfrak{F}} \left\langle A \cap^{\mathfrak{F}} \right\rangle S$ *for every $A \in \mathfrak{A}$ and every set $S \in \mathcal{P}\mathfrak{F}$.*

Proof Direct consequence of the lemma. □

8. Generalized filter base

Definition 58 ***Generalized filter base*** *is a filter base on the set $\mathfrak{F}$.*

Definition 59 *If S is a generalized filter base and $\mathcal{A} = \bigcap^{\mathfrak{F}} S$, then we will call S a generalized base of filter object $\mathcal{A}$.*

Theorem 41 *If $\mathfrak{A}$ is a distributive lattice and S is a generalized base of filter object $\mathcal{F}$ then for any element K of the base poset*

$$K \in \operatorname{up} \mathcal{F} \Leftrightarrow \exists \mathcal{L} \in S : \mathcal{L} \subseteq K.$$

Proof

$\Leftarrow$ Because $\mathcal{F} = \bigcap^{\mathfrak{F}} S$.

$\Rightarrow$ Let $K \in \operatorname{up} \mathcal{F}$. Then (taken in account distributivity of $\mathfrak{A}$ and that S is nonempty) exist $X_1, \ldots, X_n \in \bigcup \langle \operatorname{up} \rangle S$ such that $X_1 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} X_n = K$. Consequently (by theorem 29) $X_1 \cap^{\mathfrak{F}} \ldots \cap^{\mathfrak{F}} X_n = K$. Replacing every X_i with such $\mathcal{X}_i \in S$ that $X_i \in \operatorname{up} \mathcal{X}_i$ (this is obviously possible to do), we get a finite set $T_0 \subseteq S$ such that $\bigcap^{\mathfrak{F}} T_0 \subseteq K$. From this exists $\mathcal{C} \in S$ such that $\mathcal{C} \subseteq \bigcap^{\mathfrak{F}} T_0 \subseteq K$.

□

Corollary 12 *If $\mathfrak{A}$ is a distributive lattice with least element 0 and S is a generalized base of filter object $\mathcal{F}$ then $0 \in S \Leftrightarrow \mathcal{F} = 0$.*

Proof Substitute 0 as K. □

Theorem 42 *Let $\mathfrak{A}$ be a distributive lattice with least element 0 and S is a nonempty set of filter objects on $\mathfrak{A}$ such that $\mathcal{F}_0 \cap^{\mathfrak{F}} \ldots \cap^{\mathfrak{F}} \mathcal{F}_n \neq 0$ for every $\mathcal{F}_0, \ldots, \mathcal{F}_n \in S$. Then $\bigcap^{\mathfrak{F}} S \neq 0$.*

Proof Consider the set

$$S' = \left\{ \mathcal{F}_0 \cap^{\mathfrak{F}} \ldots \cap^{\mathfrak{F}} \mathcal{F}_n \mid \mathcal{F}_0, \ldots, \mathcal{F}_n \in S \right\}.$$

Obviously S' is nonempty and finitely meet-closed. So S' is a generalized filter base. Obviously $0 \notin S'$. So by properties of generalized filter bases $\bigcap^{\mathfrak{F}} S' \neq 0$. But obviously $\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} S'$. So $\bigcap^{\mathfrak{F}} S \neq 0$. □

Corollary 13 *Let $\mathfrak{A}$ be a distributive lattice with least element 0 and let $S \in \mathcal{P}\mathfrak{A}$ such that $S \neq \emptyset$ and $A_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} A_n \neq 0$ for every $A_0, \ldots, A_n \in S$. Then $\bigcap^{\mathfrak{F}} S \neq 0$.*

Proof Because $\mathfrak{A}$ is finitely meet-closed (by the theorem 29). $\square$

9. Stars

9.1. Free stars

Definition 60 *Let $\mathfrak{A}$ be a poset. **Free stars** on $\mathfrak{A}$ are such $S \in \mathcal{P}\mathfrak{A}$ that the least element (if it exists) is not in S and for every $X, Y \in \mathfrak{A}$*

$$\forall Z \in \mathfrak{A} : (Z \supseteq X \wedge Z \supseteq Y \Rightarrow Z \in S) \Leftrightarrow X \in S \vee Y \in S.$$

Proposition 25 *$S \in \mathcal{P}\mathfrak{A}$ where $\mathfrak{A}$ is a poset is a free star iff all of the following:*

1. *The least element (if it exists) is not in S.*

2. *$\forall Z \in \mathfrak{A} : (Z \supseteq X \wedge Z \supseteq Y \Rightarrow Z \in S) \Rightarrow X \in S \vee Y \in S$ for every $X, Y \in \mathfrak{A}$.*

3. *S is an upper set.*

Proof

$\Rightarrow$ (1) and (2) are obvious. Let prove that S is an upper set. Let $X \in S$ and $X \subseteq Y \in \mathfrak{A}$. Then $X \in S \vee X \in S$ and thus $\forall Z \in \mathfrak{A} : (Z \supseteq X \wedge Z \supseteq X \Rightarrow Z \in S)$ that is $\forall Z \in \mathfrak{A} : (Z \supseteq X \Rightarrow Z \in S)$, and so $Y \in S$.

$\Leftarrow$ We need to prove that

$$\forall Z \in \mathfrak{A} : (Z \supseteq X \wedge Z \supseteq Y \Rightarrow Z \in S) \Leftarrow X \in S \vee Y \in S.$$

Let $X \in S \vee Y \in S$. Then $Z \supseteq X \wedge Z \supseteq Y \Rightarrow Z \in S$ for every $Z \in \mathfrak{A}$ because S is an upper set.

$\square$

Proposition 26 *Let $\mathfrak{A}$ be a join-semilattice. $S \in \mathcal{P}\mathfrak{A}$ is a free star iff all of the following:*

1. *The least element (if it exists) is not in S.*

2. *$X \cup Y \in S \Rightarrow X \in S \vee Y \in S$ for every $X, Y \in \mathfrak{A}$.*

3. *S is an upper set.*

Proof

$\Rightarrow$ We need to prove only $X \cup Y \in S \Rightarrow X \in S \vee Y \in S$. Let $X \cup Y \in S$. Because S is an upper set, we have $\forall Z \in \mathfrak{A} : (Z \supseteq X \cup Y \Rightarrow Z \in S)$ and thus $\forall Z \in \mathfrak{A} : (Z \supseteq X \wedge Z \supseteq Y \Rightarrow Z \in S)$ from which we conclude $X \in S \vee Y \in S$.

$\Leftarrow$ We need to prove $\forall Z \in \mathfrak{A} : (Z \supseteq X \wedge Z \supseteq Y \Rightarrow Z \in S) \Leftarrow X \in S \vee Y \in S$. But this trivially follows from that S is an upper set.

□

Proposition 27 *Let $\mathfrak{A}$ be a join-semilattice. $S \in \mathcal{P}\mathfrak{A}$ is a free star iff the least element (if it exists) is not in S and for every $X, Y \in \mathfrak{A}$*

$$X \cup Y \in S \Leftrightarrow X \in S \vee Y \in S.$$

Proof

$\Rightarrow$ We need to prove only that $X \cup Y \in S \Leftarrow X \in S \vee Y \in S$ what follows from that S is an upper set.

$\Leftarrow$ We need to prove only that S is an upper set. Let $X \in S$ and $X \subseteq Y \in \mathfrak{A}$. Then $X \in S \Rightarrow X \in S \vee Y \in S \Leftrightarrow X \cup Y \in S \Rightarrow Y \in S$. So S is an upper set.

□

9.2. Stars of elements of filtrators

Definition 61 *Let $(\mathfrak{A}; \mathfrak{Z})$ be a filtrator.* ***Core star*** *of an element a of this filtrator is*

$$\partial a = \left\{ x \in \mathfrak{Z} \mid x \not\asymp^{\mathfrak{A}} a \right\}.$$

Proposition 28 $\operatorname{up} a \subseteq \partial a$ *for any non-least element a of a filtrator.*

Proof For any element $X \in \mathfrak{Z}$

$$X \in \operatorname{up} a \Rightarrow a \subseteq X \wedge a \subseteq a \Rightarrow X \not\asymp a \Rightarrow X \in \partial a.$$

□

Theorem 43 *Let $(\mathfrak{A}; \mathfrak{Z})$ be a distributive lattice filtrator with least element and finitely join-closed core which is a join-semilattice. Then ∂a is a free star for each $a \in \mathfrak{A}$.*

Proof For every $A, B \in \mathfrak{Z}$

$$\begin{aligned} A \cup^{\mathfrak{Z}} B \in \partial a &\Leftrightarrow \\ A \cup^{\mathfrak{A}} B \in \partial a &\Leftrightarrow \\ (A \cup^{\mathfrak{A}} B) \cap^{\mathfrak{A}} a \neq 0 &\Leftrightarrow \\ (A \cap^{\mathfrak{A}} a) \cup^{\mathfrak{A}} (B \cap^{\mathfrak{A}} a) \neq 0 &\Leftrightarrow \\ A \cap^{\mathfrak{A}} a \neq 0 \vee B \cap^{\mathfrak{A}} a \neq 0 &\Leftrightarrow \\ A \in \partial a \vee \in \partial a. & \end{aligned}$$

That ∂a doesn't contain 0 is obvious. □

Definition 62 *I call a filtrator* ***star-separable*** *when its core is a separation subset of its base.*

9.3. Stars of filters on boolean lattices

In this section we will consider the set of filter objects $\mathfrak{F}$ on a boolean lattice $\mathfrak{A}$.

Theorem 44 *If $\mathfrak{A}$ is a boolean lattice and $\mathcal{A} \in \mathfrak{F}$ then*

1. $\partial \mathcal{A} = \left\{ \overline{X} \mid X \in \mathfrak{A} \setminus \operatorname{up} \mathcal{A} \right\}$;

2. $\operatorname{up} \mathcal{A} = \left\{ \overline{X} \mid X \in \mathfrak{A} \setminus \partial \mathcal{A} \right\}$.

Proof 1. For any $K \in \mathfrak{A}$ (taking into account the theorems 29, 37, and 27)

$$\begin{aligned} K \in \left\{ \overline{X} \mid X \in \mathfrak{A} \setminus \operatorname{up} \mathcal{A} \right\} &\Leftrightarrow \\ \overline{K} \in \mathfrak{A} \setminus \operatorname{up} \mathcal{A} &\Leftrightarrow \\ \overline{K} \notin \operatorname{up} \mathcal{A} &\Leftrightarrow \\ \overline{K} \not\supseteq \mathcal{A} &\Leftrightarrow \\ K \not\asymp^{\mathfrak{F}} \mathcal{A} &\Leftrightarrow \\ K \in \partial \mathcal{A}. & \end{aligned}$$

2. For any $K \in \mathfrak{A}$ (taking into account the same theorems)

$$\begin{aligned} K \in \left\{ \overline{X} \mid X \in \mathfrak{A} \setminus \partial \mathcal{A} \right\} &\Leftrightarrow \\ \overline{K} \in \mathfrak{A} \setminus \partial \mathcal{A} &\Leftrightarrow \\ \overline{K} \notin \partial \mathcal{A} &\Leftrightarrow \\ \overline{K} \asymp^{\mathfrak{F}} \mathcal{A} &\Leftrightarrow \\ K \supseteq \mathcal{A} &\Leftrightarrow \\ K \in \operatorname{up} \mathcal{A}. & \end{aligned}$$

□

Corollary 14 *If $\mathfrak{A}$ is a boolean lattice, $X \in \operatorname{up}\mathcal{A} \Leftrightarrow \overline{X} \notin \partial\mathcal{A}$ for every $X \in \mathfrak{A}$, $\mathcal{A} \in \mathfrak{F}$.*

Corollary 15 *If $\mathfrak{A}$ is a boolean lattice, ∂ is an injection.*

Theorem 45 *If $\mathfrak{A}$ is a boolean lattice, then for any set $S \in \mathcal{P}\mathfrak{A}$ exists filter object $\mathcal{A}$ such that $\partial\mathcal{A} = S$ iff S is a free star.*

Proof

$\Rightarrow$ That $0 \notin S$ is obvious. For every $A, B \in \mathfrak{A}$

$$\begin{aligned} A \cup^{\mathfrak{A}} B \in S &\Leftrightarrow \\ (A \cup^{\mathfrak{A}} B) \cap^{\mathfrak{F}} \mathcal{A} \neq 0 &\Leftrightarrow \\ (A \cup^{\mathfrak{F}} B) \cap^{\mathfrak{F}} \mathcal{A} \neq 0 &\Leftrightarrow \\ (A \cap^{\mathfrak{F}} \mathcal{A}) \cup^{\mathfrak{F}} (B \cap^{\mathfrak{F}} \mathcal{A}) \neq 0 &\Leftrightarrow \\ A \cap^{\mathfrak{F}} \mathcal{A} \neq 0 \vee B \cap^{\mathfrak{F}} \mathcal{A} \neq 0 &\Leftrightarrow \\ A \in S \vee B \in S. & \end{aligned}$$

(taken into account the corollary 10 and theorem 23).

$\Leftarrow$ Let $0 \notin S$ and $\forall A, B \in S : (A \cup^{\mathfrak{A}} B \in S \Leftrightarrow A \in S \vee B \in S)$. Let $T = \left\{ \overline{X} \mid X \in \mathfrak{A} \setminus S \right\}$. We will prove that T is a filter.

$1 \in T$ because $0 \notin S$; so T is nonempty. To prove that T is a filter is enough to show that $\forall X, Y \in \mathfrak{A} : (X, Y \in T \Leftrightarrow X \cap^{\mathfrak{A}} Y \in T)$. In fact,

$$\begin{aligned} X, Y \in T &\Leftrightarrow \\ \overline{X}, \overline{Y} \notin S &\Leftrightarrow \\ \neg(\overline{X} \in S \vee \overline{Y} \in S) &\Leftrightarrow \\ \overline{X} \cup^{\mathfrak{A}} \overline{Y} \notin S &\Leftrightarrow \\ \overline{\overline{X} \cup^{\mathfrak{A}} \overline{Y}} \in T &\Leftrightarrow \\ X \cap^{\mathfrak{A}} Y \in T. & \end{aligned}$$

So T is a filter. Let $\operatorname{up}\mathcal{A} = T$ for some filter object $\mathcal{A}$.

To finish the proof we will show that $\partial\mathcal{A} = S$. In fact, for every $X \in \mathfrak{A}$

$$X \in \partial\mathcal{A} \Leftrightarrow \overline{X} \notin \operatorname{up}\mathcal{A} \Leftrightarrow \overline{X} \notin T \Leftrightarrow X \in S.$$

$\square$

Proposition 29 *If $\mathfrak{A}$ is a boolean lattice then $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \partial\mathcal{A} \subseteq \partial\mathcal{B}$ for every $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.*

Proof

$$
\begin{aligned}
\partial\mathcal{A} \subseteq \partial\mathcal{B} &\Leftrightarrow \\
\left\{ \overline{X} \mid X \in \mathfrak{A} \setminus \operatorname{up}\mathcal{A} \right\} \subseteq \left\{ \overline{X} \mid X \in \mathfrak{A} \setminus \operatorname{up}\mathcal{B} \right\} &\Leftrightarrow \\
\mathfrak{A} \setminus \operatorname{up}\mathcal{A} \subseteq \mathfrak{A} \setminus \operatorname{up}\mathcal{B} &\Leftrightarrow \\
\operatorname{up}\mathcal{A} \supseteq \operatorname{up}\mathcal{B} &\Leftrightarrow \\
\mathcal{A} \subseteq \mathcal{B}.
\end{aligned}
$$

□

Corollary 16 *∂ is a straight monotone map.*

Theorem 46 *If $\mathfrak{A}$ is a boolean lattice then $\partial \bigcup^{\mathfrak{F}} S = \bigcup \langle \partial \rangle S$.*

Proof For boolean lattices ∂ is an order embedding from the poset $\mathfrak{F}$ to the complete lattice $\mathcal{P}\mathfrak{A}$. So accordingly the lemma 2 it enough to prove that it exists $\mathcal{F} \in \mathfrak{F}$ such that $\partial\mathcal{F} = \bigcup \langle \partial \rangle S$. To prove this is enough to show that $0 \notin \bigcup \langle \partial \rangle S$ and

$$\forall A, B \in S : \left(A \cup^{\mathfrak{A}} B \in \bigcup \langle \partial \rangle S \Leftrightarrow A \in \bigcup \langle \partial \rangle S \vee B \in \bigcup \langle \partial \rangle S \right).$$

$0 \notin \bigcup \langle \partial \rangle S$ is obvious.

Let $A \cup^{\mathfrak{A}} B \in \bigcup \langle \partial \rangle S$. Then exists $Q \in \langle \partial \rangle S$ such that $A \cup^{\mathfrak{A}} B \in Q$. Then $A \in Q \vee B \in Q$, consequently $A \in \bigcup \langle \partial \rangle S \vee B \in \bigcup \langle \partial \rangle S$. Let now $A \in \bigcup \langle \partial \rangle S$. Then exists $Q \in \langle \partial \rangle S$ such that $A \in Q$, consequently $A \cup^{\mathfrak{A}} B \in Q$ and $A \cup^{\mathfrak{A}} B \in \bigcup \langle \partial \rangle S$. □

9.4. More about the lattice of filters

Theorem 47 *If $\mathfrak{A}$ is a distributive lattice with greatest element then $\mathfrak{F}$ is an atomic lattice.*

Proof Let $\mathcal{F} \in \mathfrak{F}$. Let choose (by Kuratowski's lemma) a maximal chain S from 0 to $\mathcal{F}$. Let $S' = S \setminus \{0\}$. $a = \bigcap^{\mathfrak{F}} S' \neq 0$ by properties of generalized filter bases (the corollary 12 which uses the fact that $\mathfrak{A}$ is a distributive lattice with least element). If $a \notin S$ then then the chain S can be extended adding there element a because $0 \subset a \subseteq \mathcal{X}$ for any $\mathcal{X} \in S'$ what contradicts to maximality of the chain. So $a \in S$ and consequently $a \in S'$. Obviously a is the minimal element of S'. Consequently (taking in account maximality of the chain) there are no $\mathcal{Y} \in \mathfrak{F}$ such that $0 \subset \mathcal{Y} \subset a$. So a is an atomic filter object. Obviously $a \subseteq \mathcal{F}$. □

Obvious 20 *If $\mathfrak{A}$ is a boolean lattice then $\mathfrak{F}$ is separable.*

Theorem 48 *If $\mathfrak{A}$ is a boolean lattice then $\mathfrak{F}$ is an atomistic lattice.*

Proof Because (used the theorem 20) $\mathfrak{F}$ is atomic (the theorem 47) and separable. $\square$

Corollary 17 *If $\mathfrak{A}$ is a boolean lattice then $\mathfrak{F}$ is atomically separable.*

Proof By the theorem 14. $\square$

Theorem 49 *When the base poset $\mathfrak{A}$ is a boolean lattice, then the filtrator $(\mathfrak{F}; \mathfrak{A})$ is central.*

Proof We can conclude that $\mathfrak{F}$ is atomically separable (the corollary 17) and with separable core (the theorem 37).

We need to prove that $Z(\mathfrak{F}) = \mathfrak{A}$.

Let $\mathcal{X} \in Z(\mathfrak{F})$. Then exists $\mathcal{Y} \in Z(\mathfrak{F})$ such that $\mathcal{X} \cap^{\mathfrak{F}} \mathcal{Y} = 0$ and $\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = 1$. Consequently there are $X \in \operatorname{up} \mathcal{X}$ such that $X \cap^{\mathfrak{F}} \mathcal{Y} = 0$; we have also $X \cup^{\mathfrak{F}} \mathcal{Y} = 1$. Suppose $X \supset \mathcal{X}$. Then exists $a \in \operatorname{atoms}^{\mathfrak{F}} X$ such that $a \notin \operatorname{atoms}^{\mathfrak{F}} \mathcal{X}$. We can conclude also $a \notin \operatorname{atoms}^{\mathfrak{F}} \mathcal{Y}$ (otherwise $X \cap^{\mathfrak{F}} \mathcal{Y} \neq 0$). Thus $a \notin \operatorname{atoms}^{\mathfrak{F}}(\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y})$ and consequently $\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} \neq 1$ what is a contradiction. We have $\mathcal{X} = X \in \mathfrak{A}$.

Let now $X \in \mathfrak{A}$. Let $Y = 1 \setminus^{\mathfrak{A}} X$. We have $X \cap^{\mathfrak{A}} Y = 0$ and $X \cup^{\mathfrak{A}} Y = 1$. Thus $X \cap^{\mathfrak{F}} Y = \bigcap^{\mathfrak{A}} \left\{ X \cap^{\mathfrak{A}} Y \right\} = 0$; $X \cup^{\mathfrak{F}} Y = \bigcap^{\mathfrak{F}} (\operatorname{up} X \cap \operatorname{up} Y) = \bigcap^{\mathfrak{F}} \{1\} = 1$. We have shown that $X \in Z(\mathfrak{F})$. $\square$

10. Atomic filter objects

See [2] and [4] for more detailed treatment of ultrafilters and prime filters.

Theorem 50 *Let $(\mathfrak{A}; \mathfrak{Z})$ be a semifiltered down-aligned filtrator with finitely meet-closed core $\mathfrak{Z}$ which is a meet-semilattice. Then a is an atom of $\mathfrak{Z}$ iff $a \in \mathfrak{Z}$ and a is an atom of $\mathfrak{A}$.*

Proof

$\Leftarrow$ Obvious.

$\Rightarrow$ We need to prove that if a is an atom of $\mathfrak{Z}$ then a is an atom of $\mathfrak{A}$. Suppose the contrary that a is not an atom of $\mathfrak{A}$. Then exists $x \in \mathfrak{A}$ such that $0 \neq x \subset a$. Because "up" is a straight monotone map from $\mathfrak{A}$ to the dual of the poset $\mathcal{P}\mathfrak{Z}$ (the theorem 10), $\operatorname{up} a \subset \operatorname{up} x$. So exists $K \in \operatorname{up} x$ such that $K \notin \operatorname{up} a$. Also $a \in \operatorname{up} x$. We have $K \cap^{\mathfrak{Z}} a = K \cap^{\mathfrak{A}} a \in \operatorname{up} x$; $K \cap^{\mathfrak{Z}} a \neq 0$ and $K \cap^{\mathfrak{Z}} a \subset a$. So a is not an atom of $\mathfrak{Z}$.

$\square$

Theorem 51 *Let $(\mathfrak{A}; \mathfrak{Z})$ be a down-aligned semifiltered filtrator and $\mathfrak{A}$ is a meet-semilattice. Then $a \in \mathfrak{A}$ is an atom of $\mathfrak{A}$ iff $\operatorname{up} a = \partial a$.*

Proof

$\Rightarrow$ Let a be an atom of $\mathfrak{A}$. $\operatorname{up} a \supseteq \partial a$ because $a \neq 0$. $\operatorname{up} a \subseteq \partial a$ because for any $K \in \mathfrak{A}$

$$K \in \operatorname{up} a \Leftrightarrow K \supseteq a \Rightarrow K \cap^{\mathfrak{A}} a \neq 0 \Leftrightarrow K \in \partial a.$$

$\Leftarrow$ Let $\operatorname{up} a = \partial a$. Then $a \neq 0$. Consequently for every $x \in \mathfrak{A}$ we have

$$\begin{aligned} 0 \subset x \subset a &\Rightarrow \\ x \cap^{\mathfrak{A}} a \neq 0 &\Rightarrow \\ \forall K \in \operatorname{up} x : K \in \partial a &\Rightarrow \\ \forall K \in \operatorname{up} x : K \in \operatorname{up} a &\Rightarrow \\ \operatorname{up} x \subseteq \operatorname{up} a &\Rightarrow \\ x \supseteq a. & \end{aligned}$$

So a is an atom of $\mathfrak{A}$.

□

10.1. Prime filtrator elements

Definition 63 *Let $(\mathfrak{A}; \mathfrak{Z})$ be a down-aligned filtrator with least element 0.* ***Prime*** *filtrator elements are such $a \in \mathfrak{A}$ that $\operatorname{up} a$ is a free star.*

Proposition 30 *Let $(\mathfrak{A}; \mathfrak{Z})$ be a down-aligned filtrator with finitely join-closed core, where $\mathfrak{A}$ is a distributive lattice and $\mathfrak{Z}$ is a join-semilattice. Then atomic elements of this filtrator are prime.*

Proof Let a be an atom of the lattice $\mathfrak{A}$. We have for every $X, Y \in \mathfrak{Z}$

$$\begin{aligned} X \cup^{\mathfrak{Z}} Y \in \operatorname{up} a &\Leftrightarrow \\ X \cup^{\mathfrak{A}} Y \in \operatorname{up} a &\Leftrightarrow \\ X \cup^{\mathfrak{A}} Y \supseteq a &\Leftrightarrow \\ (X \cup^{\mathfrak{A}} Y) \cap^{\mathfrak{A}} a \neq 0 &\Leftrightarrow \\ (X \cap^{\mathfrak{A}} a) \cup^{\mathfrak{A}} (Y \cap^{\mathfrak{A}} a) \neq 0 &\Leftrightarrow \\ X \cap^{\mathfrak{A}} a \neq 0 \vee Y \cap^{\mathfrak{A}} a \neq 0 &\Leftrightarrow \\ X \supseteq a \vee Y \supseteq a &\Leftrightarrow \\ X \in \operatorname{up} a \vee Y \in \operatorname{up} a. & \end{aligned}$$

□

The following theorem is essentially borrowed from [8]:

Theorem 52 *Let $\mathfrak{A}$ be a boolean lattice. Let a be a f.o. Then the following are equivalent:*

1. a is prime.

2. For every $A \in \mathfrak{A}$ exactly one of $\left\{A, \overline{A}\right\}$ is in $\operatorname{up} a$.

3. a is an atom of $\mathfrak{F}$.

Proof

(1)⇒(2) Let a be prime. Then $A \cup^{\mathfrak{A}} \overline{A} = 1 \in \operatorname{up} a$. Therefore $A \in \operatorname{up} a \vee \overline{A} \in \operatorname{up} a$. But since $A \cap^{\mathfrak{A}} \overline{A} = 0 \notin \operatorname{up} a$ it is impossible $A \in \operatorname{up} a \wedge \overline{A} \in \operatorname{up} a$.

(2)⇒(3) Obviously $a \neq 0$. Let f.o. $b \subset a$. So $\operatorname{up} b \supset \operatorname{up} a$. Let $X \in \operatorname{up} b \setminus \operatorname{up} a$. Then $X \notin \operatorname{up} a$ and thus $\overline{X} \in \operatorname{up} a$ and consequently $\overline{X} \in \operatorname{up} b$. So $0 = X \cap^{\mathfrak{A}} \overline{X} \in \operatorname{up} b$ and thus $b = 0$. So a is atomic.

(3)⇒(1) By the previous proposition (taking into account the corollary 10 and the theorem 23).

□

11. Some criteria

Theorem 53 *For a semifiltered, star-separable, down-aligned filtrator* $(\mathfrak{A}; \mathfrak{Z})$ *with finitely meet closed and separable core where $\mathfrak{Z}$ is a complete boolean lattice and both $\mathfrak{Z}$ and $\mathfrak{A}$ are atomistic lattices the following conditions are equivalent for any $\mathcal{F} \in \mathfrak{A}$:*

1. $\mathcal{F} \in \mathfrak{Z}$;

2. $\forall S \in \mathscr{P}\mathfrak{A} : (\mathcal{F} \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} S \neq 0 \Rightarrow \exists \mathcal{K} \in S : \mathcal{F} \cap^{\mathfrak{A}} \mathcal{K} \neq 0)$;

3. $\forall S \in \mathscr{P}\mathfrak{Z} : (\mathcal{F} \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} S \neq 0 \Rightarrow \exists K \in S : \mathcal{F} \cap^{\mathfrak{A}} K \neq 0)$.

Proof Our filtrator is with join-closed core.

(1)⇒(2) Let $\mathcal{F} \in \mathfrak{Z}$. Then (taking in account the proposition 27)

$$\mathcal{F} \cap^{\mathfrak{A}} \bigcup\nolimits^{\mathfrak{A}} S \neq 0 \Leftrightarrow \overline{\mathcal{F}} \not\supseteq \bigcup\nolimits^{\mathfrak{A}} S \Rightarrow \exists \mathcal{K} \in S : \overline{\mathcal{F}} \not\supseteq \mathcal{K} \Leftrightarrow \exists \mathcal{K} \in S : \mathcal{F} \cap^{\mathfrak{A}} \mathcal{K} \neq 0.$$

(2)⇒(3) Obvious.

(3)⇒(1) Let the formula (3) be true. Then for $L \in \mathfrak{Z}$ and $S = \operatorname{atoms}^{\mathfrak{Z}} L$ it takes the form $\mathcal{F} \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} \operatorname{atoms}^{\mathfrak{Z}} L \neq 0 \Rightarrow \exists K \in S : \mathcal{F} \cap^{\mathfrak{A}} K \neq 0$ that is $\mathcal{F} \cap^{\mathfrak{A}} L \neq 0 \Rightarrow \exists K \in S : \mathcal{F} \cap^{\mathfrak{A}} K \neq 0$ because $\bigcup^{\mathfrak{A}} \operatorname{atoms}^{\mathfrak{Z}} L = \bigcup^{\mathfrak{Z}} \operatorname{atoms}^{\mathfrak{Z}} L = L$. That is $\mathcal{F} \cap^{\mathfrak{A}} L \neq 0 \Rightarrow \mathcal{F} \cap^{\mathfrak{A}} K_L \neq 0$ where $K_L \in S$. Thus K_L is an atom of both $\mathfrak{A}$ and $\mathfrak{Z}$ (see the theorem 50), so having $\mathcal{F} \cap^{\mathfrak{A}} L \neq 0 \Rightarrow \mathcal{F} \supseteq K_L$. Let

$$F = \bigcup\nolimits^{\mathfrak{Z}} \left\{ K_L \mid L \in \mathfrak{Z}, \mathcal{F} \cap^{\mathfrak{A}} L \neq 0 \right\}.$$

Then

$$F = \bigcup\nolimits^{\mathfrak{A}} \left\{ K_L \mid L \in \mathfrak{Z}, \mathcal{F} \cap^{\mathfrak{A}} L \neq 0 \right\}.$$

Obviously $F \subseteq \mathcal{F}$. We have $L \cap^{\mathfrak{A}} \mathcal{F} \neq 0 \Rightarrow L \cap^{\mathfrak{Z}} \mathcal{F} \neq 0 \Rightarrow L \cap^{\mathfrak{A}} \mathcal{F} \neq 0 \Rightarrow K_L \cap^{\mathfrak{Z}} F \neq 0 \Rightarrow L \cap^{\mathfrak{Z}} F \neq 0$, thus by star separability of our filtrator $\mathcal{F} \subseteq F$ and so $\mathcal{F} = F \in \mathfrak{Z}$.

□

Theorem 54 *If $\mathfrak{A}$ is a complete boolean lattice then for each $\mathcal{F} \in \mathfrak{F}$*

$$\mathcal{F} \in \mathfrak{A} \Leftrightarrow \forall S \in \mathcal{P}\mathfrak{A} : \left(\bigcup\nolimits^{\mathfrak{A}} S \in \partial \mathcal{F} \Rightarrow S \cap \partial \mathcal{F} \neq \emptyset \right).$$

Proof

$$\begin{aligned}
\forall S \in \mathcal{P}\mathfrak{A} : \left(\bigcup\nolimits^{\mathfrak{A}} S \in \partial \mathcal{F} \Rightarrow S \cap \partial \mathcal{F} \neq \emptyset \right) \quad & \Leftrightarrow \\
\forall S \in \mathcal{P}\mathfrak{A} : \left(\bigcup\nolimits^{\mathfrak{A}} S \notin \partial \mathcal{F} \Leftarrow S \cap \partial \mathcal{F} = \emptyset \right) \quad & \Leftrightarrow \\
\forall S \in \mathcal{P}\mathfrak{A} : \left(\overline{\bigcup\nolimits^{\mathfrak{A}} S} \in \operatorname{up} \mathcal{F} \Leftarrow \langle \neg \rangle S \subseteq \operatorname{up} \mathcal{F} \right) \quad & \Leftrightarrow \\
\forall S \in \mathcal{P}\mathfrak{A} : \left(\bigcap\nolimits^{\mathfrak{A}} S \in \operatorname{up} \mathcal{F} \Leftarrow S \subseteq \operatorname{up} \mathcal{F} \right), &
\end{aligned}$$

but

$$\begin{aligned}
\mathcal{F} \in \mathfrak{A} \quad & \Rightarrow \\
\forall S \in \mathcal{P}\mathfrak{A} : \left(\bigcap\nolimits^{\mathfrak{A}} S \in \operatorname{up} \mathcal{F} \Leftarrow S \subseteq \operatorname{up} \mathcal{F} \right) \quad & \Rightarrow \\
\bigcap\nolimits^{\mathfrak{A}} \operatorname{up} \mathcal{F} \in \operatorname{up} \mathcal{F} \quad & \Rightarrow \\
\mathcal{F} \in \mathfrak{A}. &
\end{aligned}$$

□

Definition 64 *Let S be a subset of a meet-semilattice. The **filter base generated by** S is the set*

$$[S]_\cap \stackrel{\text{def}}{=} \{ a_0 \cap \ldots \cap a_n \mid a_i \in S, i = 0, 1, \ldots \}.$$

Lemma 4 *The set of all finite subsets of an infinite set A has the same cardinality as A.*

Proof Let denote the number of n-element subsets of A as s_n. Obviously $s_n \leqslant \operatorname{card} A^n = \operatorname{card} A$. Then the number S of all finite subsets of A is equal to $s_0 + s_1 + \ldots \leqslant \operatorname{card} A + \operatorname{card} A + \ldots = \operatorname{card} A$. That $S \geqslant \operatorname{card} A$ is obvious. So $S = \operatorname{card} A$. □

Lemma 5 *A filter base generated by an infinite set has the same cardinality as that set.*

Proof From the previous lemma. □

Definition 65 *Let $\mathfrak{A}$ be a complete lattice. A set $S \in \mathscr{P}\mathfrak{A}$ is* ***filter-closed*** *when for every filter base $T \in \mathscr{P}S$ we have $\bigcap T \in S$.*

Theorem 55 *A subset S of a complete lattice is filter-closed iff for every nonempty chain $T \in \mathscr{P}S$ we have $\bigcap T \in S$.*

Proof (proof sketch by Joel David Hamkins)

$\Rightarrow$ Because every nonempty chain is a filter base.

$\Leftarrow$ We will assume that cardinality of a set is an ordinal defined by von Neumann cardinal assignment (what is a standard practice in ZFC). Recall that $\alpha < \beta \Leftrightarrow \alpha \in \beta$ for ordinals α, β.

We will take it as given that for every nonempty chain $T \in \mathscr{P}S$ we have $\bigcap T \in S$.

We will prove the following statement: If $\operatorname{card} S = n$ then S is filter closed, for any cardinal n.

Instead we will prove it not only for cardinals but for wider class of ordinals: If $\operatorname{card} S = n$ then S is filter closed, for any ordinal n.

We will prove it using transfinite induction by n.

For finite n we have $\bigcap T \in S$ because $T \subseteq S$ has minimal element.

Let $\operatorname{card} T = n$ be an infinite ordinal.

Let the assumption of induction holds for every $n \in \operatorname{card} T$.

We can assign $T = \{a_\alpha \mid \alpha \in \operatorname{card} T\}$ for some a_α because $\operatorname{card}\operatorname{card} T = \operatorname{card} T$.

Consider $\beta \in \operatorname{card} T$.

Let $P_\beta = \{a_\alpha \mid \alpha \in \beta\}$. Let $b_\beta = \bigcap P_\beta$. Obviously $b_\beta = \bigcap [P_\beta]_\cap$. We have

$$\operatorname{card}[P_\beta]_\cap = \operatorname{card} P_\beta = \operatorname{card} \beta < \operatorname{card} T$$

(used the lemma and von Neumann cardinal assignment). By the assumption of induction $b_\beta \in S$.

$\forall \beta \in \operatorname{card} T : P_\beta \subseteq T$ and thus $b_\beta \supseteq \bigcap T$.

Easy to see that the set $\{P_\beta \mid \beta \in \operatorname{card} T\}$ is a chain. Consequently $\{b_\beta \mid \beta \in \operatorname{card} T\}$ is a chain.

By theorem conditions $b = \bigcap \{b_\beta \mid \beta \in \operatorname{card} T\} \in S$ (taken in account that $b_\beta \in S$).

Obviously $b \supseteq \bigcap T$.

$b \subseteq b_\beta$ and so $\forall \beta \in \operatorname{card} T \forall \alpha \in \beta : b \subseteq a_\alpha$. Let $\alpha \in \operatorname{card} T$. Then (because $\operatorname{card} A$ is limit ordinal, see [15]) exist $\beta \in \operatorname{card} T$ such that $\alpha \in \beta \in \operatorname{card} T$. So $b \subseteq a_\alpha$ for every $\alpha \in \operatorname{card} T$. Thus $b \subseteq \bigcap T$.

Finally $\bigcap T = b \in S$.

□

Theorem 56 *Let $\mathfrak{A}$ be a boolean lattice. For any $S \in \mathcal{P}\mathfrak{F}$ the condition $\exists \mathcal{F} \in \mathfrak{F} : S = \star\mathcal{F}$ is equivalent to conjunction of the following items:*

1. *S is a free star on $\mathfrak{F}$;*
2. *S is filter-closed.*

Proof

$\Rightarrow$ 1. That $0 \notin \star\mathcal{F}$ is obvious. For every $a, b \in \mathfrak{F}$

$$\begin{aligned} a \cup^{\mathfrak{F}} b \in \star\mathcal{F} &\Leftrightarrow \\ (a \cup^{\mathfrak{F}} b) \cap^{\mathfrak{F}} \mathcal{F} \neq 0 &\Leftrightarrow \\ (a \cap^{\mathfrak{F}} \mathcal{F}) \cup (b \cap^{\mathfrak{F}} \mathcal{F}) \neq 0 &\Leftrightarrow \\ a \cap^{\mathfrak{F}} \mathcal{F} \neq 0 \vee b \cap^{\mathfrak{F}} \mathcal{F} \neq 0 &\Leftrightarrow \\ a \in \star S \vee b \in \star\mathcal{F}. \end{aligned}$$

(taken into account the corollary 10). So $\star\mathcal{F}$ is a free star on $\mathfrak{F}$.

2. We have $T \subseteq S$ and need to prove that $\bigcap^{\mathfrak{F}} T \cap \mathcal{F} \neq 0$. Because $\left\langle \mathcal{F} \cap^{\mathfrak{F}} \right\rangle T$ is a generalized filter base, $0 \in \left\langle \mathcal{F} \cap^{\mathfrak{F}} \right\rangle T \Leftrightarrow \bigcap^{\mathfrak{F}} \left\langle \mathcal{F} \cap^{\mathfrak{F}} \right\rangle T = 0 \Leftrightarrow \bigcap^{\mathfrak{F}} T \cap^{\mathfrak{F}} \mathcal{F} = 0$. So it's left to prove $0 \notin \left\langle \mathcal{F} \cap^{\mathfrak{F}} \right\rangle T$ what follows from $T \subseteq S$.

$\Leftarrow$ Let S be a free star on $\mathfrak{F}$. Then for every $A, B \in \mathfrak{A}$

$$\begin{aligned} A, B \in S \cap \mathfrak{A} &\Leftrightarrow \\ A, B \in S &\Leftrightarrow \\ A \cup^{\mathfrak{F}} B \in S &\Leftrightarrow \\ A \cup^{\mathfrak{A}} B \in S &\Leftrightarrow \\ A \cup^{\mathfrak{A}} B \in S \cap \mathfrak{A} \end{aligned}$$

(taken into account the theorem 23). So $S \cap \mathfrak{A}$ is a free star on $\mathfrak{A}$.

Thus there exists $\mathcal{F} \in \mathfrak{F}$ such that $\partial\mathcal{F} = S \cap \mathfrak{A}$. We have $\operatorname{up} \mathcal{X} \subseteq S \Leftrightarrow \mathcal{X} \in S$ (because S is filter-closed) for every $\mathcal{X} \in \mathfrak{F}$; then (taking in account properties of generalized filter bases)

$$
\begin{aligned}
\mathcal{X} \in S &\Leftrightarrow \\
\forall X \in \operatorname{up} \mathcal{X} : X \cap^{\mathfrak{F}} \mathcal{F} \neq 0 &\Leftrightarrow \\
0 \notin \left\langle \mathcal{F} \cap^{\mathfrak{F}} \right\rangle \operatorname{up} \mathcal{X} &\Leftrightarrow \\
\bigcap{}^{\mathfrak{F}} \left\langle \mathcal{F} \cap^{\mathfrak{F}} \right\rangle \operatorname{up} \mathcal{X} \neq 0 &\Leftrightarrow \\
\mathcal{F} \cap^{\mathfrak{F}} \bigcap{}^{\mathfrak{F}} \operatorname{up} \mathcal{X} \neq 0 &\Leftrightarrow \\
\mathcal{F} \cap^{\mathfrak{F}} \mathcal{X} \neq 0 &\Leftrightarrow \\
\mathcal{X} \in \star \mathcal{F}. &
\end{aligned}
$$

□

12. Quasidifference and quasicomplement

I've got quasidifference and quasicomplement (and dual quasicomplement) replacing max and min in the definition of pseudodifference and pseudocomplement (and dual pseudocomplement) with $\bigcup$ and $\bigcap$. Thus quasidifference and (dual) quasicomplement are generalizations of their pseudo- counterparts.

Remark 11 **Pseudocomplements** and **pseudodifferences** is standard terminology. **Quasi-** counterparts are my neologisms.

Definition 66 *Let $\mathfrak{A}$ be a poset, $a \in \mathfrak{A}$.* ***Quasicomplement*** *of a is*

$$a^* = \bigcup \{ c \in \mathfrak{A} \mid c \asymp a \} .$$

Definition 67 *Let $\mathfrak{A}$ be a poset, $a \in \mathfrak{A}$.* ***Dual quasicomplement*** *of a is*

$$a^+ = \bigcap \{ c \in \mathfrak{A} \mid c \equiv a \} .$$

I will denote quasicomplement and dual quasicomplement for a specific poset $\mathfrak{A}$ as $a^{*(\mathfrak{A})}$ and $a^{+(\mathfrak{A})}$.

Definition 68 *Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a distributive lattice.* ***Quasidifference*** *of a and b is*

$$a \setminus^* b = \bigcap \{ z \in \mathfrak{A} \mid a \subseteq b \cup z \} .$$

Definition 69 *Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a distributive lattice.* ***Second quasidifference*** *of a and b is*

$$a \# b \stackrel{\text{def}}{=} \bigcup \{ z \in \mathfrak{A} \mid z \subseteq a \wedge z \asymp b \} .$$

Theorem 57 *$a \setminus^* b = \bigcap \{ z \in \mathfrak{A} \mid z \subseteq a \wedge a \subseteq b \cup z \}$ where $\mathfrak{A}$ is a distributive lattice and $a, b \in \mathfrak{A}$.*

Proof Obviously $\{ z \in \mathfrak{A} \mid z \subseteq a \wedge a \subseteq b \cup z \} \subseteq \{ z \in \mathfrak{A} \mid a \subseteq b \cup z \}$. Thus $\bigcap \{ z \in \mathfrak{A} \mid z \subseteq a \wedge a \subseteq b \cup z \} \supseteq a \setminus^* b$.

Let $z \in \mathfrak{A}$ and $z' = z \cap a$.

$a \subseteq b \cup z \Rightarrow a \subseteq (b \cup z) \cap a \Leftrightarrow a \subseteq (b \cap a) \cup (z \cap a) \Leftrightarrow a \subseteq (b \cap a) \cup z' \Rightarrow a \subseteq b \cup z'$ and $a \subseteq b \cup z \Leftarrow a \subseteq b \cup z'$. Thus $a \subseteq b \cup z \Leftrightarrow a \subseteq b \cup z'$.

If $z \in \{ z \in \mathfrak{A} \mid a \subseteq b \cup z \}$ then $a \subseteq b \cup z'$ and thus $z' \in \{ z \in \mathfrak{A} \mid z \subseteq a \wedge a \subseteq b \cup z \}$. But $z' \subseteq z$ thus having $\bigcap \{ z \in \mathfrak{A} \mid z \subseteq a \wedge a \subseteq b \cup z \} \subseteq \bigcap \{ z \in \mathfrak{A} \mid a \subseteq b \cup z \}$.

□

Remark 12 If we drop the requirement that $\mathfrak{A}$ is distributive, two formulas for quasidifference (the definition and the last theorem) fork.

Obvious 21 *Dual quasicomplement is the dual of quasicomplement.*

Obvious 22

- *Every pseudocomplement is quasicomplement.*
- *Every dual pseudocomplement is dual quasicomplement.*
- *Every pseudodifference is quasidifference.*

Below we will stick to the more general quasies than pseudos. If needed, one can check that a quasicomplement a^* is a pseudocomplement by the equation $a^* \asymp a$ (and analogously with other quasies).

Next we will express quasidifference through quasicomplement.

Proposition 31

1. *$a \setminus^* b = a \setminus^* (a \cap b)$ for any distributive lattice;*
2. *$a \# b = a \# (a \cap b)$ for any distributive lattice with least element.*

Proof

1. $a \subseteq (a \cap b) \cup z \Leftrightarrow a \subseteq (a \cup z) \cap (b \cup z) \Leftrightarrow a \subseteq a \cup z \wedge a \subseteq b \cup z \Leftrightarrow a \subseteq b \cup z$. Thus $a \setminus^* (a \cap b) = \bigcap \{ z \in \mathfrak{A} \mid a \subseteq (a \cap b) \cup z \} = \bigcap \{ z \in \mathfrak{A} \mid a \subseteq b \cup z \} = a \setminus^* b$.
2. $a \# (a \cap b) = \bigcup \{ z \in \mathfrak{A} \mid z \subseteq a \wedge z \cap a \cap b = 0 \} = \bigcup \{ z \in \mathfrak{A} \mid z \subseteq a \wedge (z \cap a) \cap a \cap b = 0 \} = \bigcup \{ z \cap a \mid z \in \mathfrak{A}, z \cap a \cap b = 0 \} = \bigcup \{ z \in \mathfrak{A} \mid z \subseteq a, z \cap b = 0 \} = a \# b$.

□

I will denote Da the lattice $\{ x \in \mathfrak{A} \mid x \subseteq a \}$.

Theorem 58 *For $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a distributive lattice with least element*

1. *$a \setminus^* b = (a \cap b)^{+(Da)}$;*

2. $a\#b = (a \cap b)^{*(Da)}$.

Proof

1.

$$\begin{aligned} (a \cap b)^{+(Da)} &= \\ \bigcap \{ c \in Da \mid c \cup (a \cap b) = a \} &= \\ \bigcap \{ c \in Da \mid c \cup (a \cap b) \supseteq a \} &= \\ \bigcap \{ c \in Da \mid (c \cup a) \cap (c \cup b) \supseteq a \} &= \\ \bigcap \{ c \in \mathfrak{A} \mid c \subseteq a \wedge c \cup b \supseteq a \} &= \\ a \setminus^* b. \end{aligned}$$

2.

$$\begin{aligned} (a \cap b)^{*(Da)} &= \\ \bigcup \{ c \in Da \mid c \cap a \cap b = 0 \} &= \\ \bigcup \{ c \in \mathfrak{A} \mid c \subseteq a \wedge c \cap a \cap b = 0 \} &= \\ \bigcup \{ c \in \mathfrak{A} \mid c \subseteq a \wedge c \cap b = 0 \} &= \\ a\#b. \end{aligned}$$

□

Theorem 59 *Let $(\mathfrak{F}; \mathfrak{A})$ be a primary filtrator where $\mathfrak{A}$ is a boolean lattice. Let $\mathcal{A} \in \mathfrak{F}$. Then for each $\mathcal{X} \in \mathfrak{F}$*

$$\mathcal{X} \in Z(D\mathcal{A}) \Leftrightarrow \exists X \in \mathfrak{A} : \mathcal{X} = X \cap^{\mathfrak{F}} \mathcal{A}.$$

Proof

$\Leftarrow$ Let $\mathcal{X} = X \cap^{\mathfrak{F}} \mathcal{A}$ where $X \in \mathfrak{A}$. Let also $\mathcal{Y} = \overline{X} \cap^{\mathfrak{F}} \mathcal{A}$. Then $\mathcal{X} \cap^{\mathfrak{F}} \mathcal{Y} = X \cap^{\mathfrak{F}} \overline{X} \cap^{\mathfrak{F}} \mathcal{A} = (X \cap^{\mathfrak{A}} \overline{X}) \cap^{\mathfrak{F}} \mathcal{A} = 0$ (used the theorem 29 and $\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = (X \cup^{\mathfrak{F}} \overline{X}) \cap^{\mathfrak{F}} \mathcal{A} = (X \cup^{\mathfrak{A}} \overline{X}) \cap^{\mathfrak{F}} \mathcal{A} = 1 \cap^{\mathfrak{F}} \mathcal{A} = \mathcal{A}$ (used the theorems 23 and corollary 10). So $\mathcal{X} \in Z(D\mathcal{A})$.

$\Rightarrow$ Let $\mathcal{X} \in Z(D\mathcal{A})$. Then exists $\mathcal{Y} \in Z(D\mathcal{A})$ such that $\mathcal{X} \cap^{\mathfrak{F}} \mathcal{Y} = 0$ and $\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = \mathcal{A}$. Then (used the theorem 37) exists $X \in \operatorname{up} \mathcal{X}$ such that $X \cap^{\mathfrak{F}} \mathcal{Y} = 0$. We have

$$\mathcal{X} = \mathcal{X} \cup^{\mathfrak{F}} (X \cap^{\mathfrak{F}} \mathcal{Y}) = X \cap^{\mathfrak{F}} (\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y}) = X \cap^{\mathfrak{F}} \mathcal{A}.$$

□

Proposition 32 *$(a \cup b) \setminus^* b \subseteq a$ for an arbitrary complete lattice.*

Proof $(a \cup b) \setminus^* b = \bigcap \{z \in \mathfrak{A} \mid a \cup b \subseteq b \cup z\}$.

But $a \subseteq z \Rightarrow a \cup b \subseteq b \cup z$. So $\{z \in \mathfrak{A} \mid a \cup b \subseteq b \cup z\} \supseteq \{z \in \mathfrak{A} \mid a \subseteq z\}$.

Consequently, $(a \cup b) \setminus^* b \subseteq \bigcap \{z \in \mathfrak{A} \mid a \subseteq z\} = a$. □

13. Complements and core parts

Lemma 6 *If $(\mathfrak{A}; \mathfrak{Z})$ is a filtered, up-aligned filtrator with co-separable core which is a complete lattice, then for any $a, c \in \mathfrak{A}$*

$$c \equiv^{\mathfrak{A}} a \Leftrightarrow c \equiv^{\mathfrak{A}} \operatorname{Cor} a.$$

Proof

$\Rightarrow$ If $c \equiv^{\mathfrak{A}} a$ then by co-separability of the core exists $K \in \operatorname{down} a$ such that $c \equiv^{\mathfrak{A}} K$. To finish the proof we will show that $K \subseteq \operatorname{Cor} a$. To show this is enough to show that $\forall X \in \operatorname{up} a : K \subseteq X$ what is obvious.

$\Leftarrow$ Because $\operatorname{Cor} a \subseteq a$ (by the theorem 24 using that our filtrator is filtered).

□

Theorem 60 *If $(\mathfrak{A}; \mathfrak{Z})$ is a filtered up-aligned complete lattice filtrator with co-separable core which is a complete boolean lattice, then $a^+ = \overline{\operatorname{Cor} a}$ for every $a \in \mathfrak{A}$.*

Proof Our filtrator is with join-closed core.

$$\begin{aligned} a^+ &= \\ \bigcap^{\mathfrak{A}} \left\{c \in \mathfrak{A} \mid c \cup^{\mathfrak{A}} a = 1\right\} &= \\ \bigcap^{\mathfrak{A}} \left\{c \in \mathfrak{A} \mid c \cup^{\mathfrak{A}} \operatorname{Cor} a = 1\right\} &= \\ \bigcap^{\mathfrak{A}} \left\{c \in \mathfrak{A} \mid c \supseteq \overline{\operatorname{Cor} a}\right\} &= \\ \overline{\operatorname{Cor} a}. \end{aligned}$$

(used the lemma and the theorem 27). □

Corollary 18 *If $(\mathfrak{A}; \mathfrak{Z})$ is a filtered up-aligned complete lattice filtrator with co-separable core which is a complete boolean lattice, then $a^+ \in \mathfrak{Z}$ for every $a \in \mathfrak{A}$.*

Theorem 61 *If $(\mathfrak{A}; \mathfrak{Z})$ is a filtered complete lattice filtrator with down-aligned, finitely meet-closed, separable core which is a complete boolean lattice, then $a^* = \overline{\operatorname{Cor} a} = \overline{\operatorname{Cor}' a}$.*

Proof Our filtrator is with join-closed core. $a^* = \bigcup^{\mathfrak{A}} \left\{ c \in \mathfrak{A} \mid c \cap^{\mathfrak{A}} a = 0 \right\}$. But $c \cap^{\mathfrak{A}} a = 0 \Rightarrow \exists C \in \operatorname{up} c : C \cap^{\mathfrak{A}} a = 0$. So

$$\begin{aligned}
a^* &= \\
\bigcup\nolimits^{\mathfrak{A}} \left\{ C \in \mathfrak{Z} \mid C \cap^{\mathfrak{A}} a = 0 \right\} &= \\
\bigcup\nolimits^{\mathfrak{A}} \left\{ C \in \mathfrak{Z} \mid a \subseteq \overline{C} \right\} &= \\
\bigcup\nolimits^{\mathfrak{A}} \left\{ \overline{C} \mid C \in \mathfrak{Z}, a \subseteq C \right\} &= \\
\bigcup\nolimits^{\mathfrak{A}} \left\{ \overline{C} \mid C \in \operatorname{up} a \right\} &= \\
\bigcup\nolimits^{\mathfrak{Z}} \left\{ \overline{C} \mid C \in \operatorname{up} a \right\} &= \\
\overline{\bigcap\nolimits^{\mathfrak{Z}} \left\{ C \mid C \in \operatorname{up} a \right\}} &= \\
\overline{\bigcap\nolimits^{\mathfrak{Z}} \operatorname{up} a} &= \\
\overline{\operatorname{Cor} a}.
\end{aligned}$$

(used the theorem 27).

$\operatorname{Cor} a = \operatorname{Cor}' a$ by the theorem 26. □

Corollary 19 *If $(\mathfrak{A}; \mathfrak{Z})$ is filtered down-aligned and up-aligned complete lattice filtrator with finitely meet-closed, separable and co-separable core which is a complete boolean lattice, then $a^* = a^+$ for every $a \in \mathfrak{A}$.*

Proof Comparing two last theorems. □

Theorem 62 *If $(\mathfrak{A}; \mathfrak{Z})$ is a complete lattice filtrator with join-closed separable core which is a complete lattice, then $a^* \in \mathfrak{Z}$ for every $a \in \mathfrak{A}$.*

Proof $\left\{ c \in \mathfrak{A} \mid c \cap^{\mathfrak{A}} a = 0 \right\} \supseteq \left\{ A \in \mathfrak{Z} \mid A \cap^{\mathfrak{A}} a = 0 \right\}$; consequently $a^* \supseteq \bigcup^{\mathfrak{A}} \left\{ A \in \mathfrak{Z} \mid A \cap^{\mathfrak{A}} a = 0 \right\}$.

But if $c \in \left\{ c \in \mathfrak{A} \mid c \cap^{\mathfrak{A}} a = 0 \right\}$ then exists $A \in \mathfrak{Z}$ such that $A \supseteq c$ and $A \cap^{\mathfrak{A}} a = 0$ that is $A \in \left\{ A \in \mathfrak{Z} \mid A \cap^{\mathfrak{A}} a = 0 \right\}$. Consequently $a^* \subseteq \bigcup^{\mathfrak{A}} \left\{ A \in \mathfrak{Z} \mid A \cap^{\mathfrak{A}} a = 0 \right\}$.

We have $a^* = \bigcup^{\mathfrak{A}} \left\{ A \in \mathfrak{Z} \mid A \cap^{\mathfrak{A}} a = 0 \right\} = \bigcup^{\mathfrak{Z}} \left\{ A \in \mathfrak{Z} \mid A \cap^{\mathfrak{A}} a = 0 \right\} \in \mathfrak{Z}$. □

Theorem 63 *If $(\mathfrak{A}; \mathfrak{Z})$ is an up-aligned filtered complete lattice filtrator co-separable core which is a complete boolean lattice, then a^+ is dual pseudocomplement of a, that is $a^+ = \min \left\{ c \in \mathfrak{A} \mid c \cup^{\mathfrak{A}} a = 1 \right\}$ for every $a \in \mathfrak{A}$.*

Proof Our filtrator is with join-closed core. It's enough to prove that $a^+ \cup^{\mathfrak{A}} a = 1$. But $a^+ \cup^{\mathfrak{A}} a = \overline{\operatorname{Cor} a} \cup^{\mathfrak{A}} a \supseteq \overline{\operatorname{Cor} a} \cup^{\mathfrak{A}} \operatorname{Cor} a = \overline{\operatorname{Cor} a} \cup^{\mathfrak{Z}} \operatorname{Cor} a = 1$ (used the theorem 24 and the fact that our filtrator is filtered). □

Definition 70 *The **edge part** of an element* $a \in \mathfrak{A}$ *is* $\operatorname{Edg} a = a \setminus \operatorname{Cor} a$*, the **dual edge part** is* $\operatorname{Edg}' a = a \setminus \operatorname{Cor}' a$*.*

Proposition 33 *For a primary filtrator over a complete boolean lattice both edge part and dual edge part are always defined.*

Proof Using the theorem 39. □

Knowing core part and edge part or dual core part and dual edge part of a filter object, the filter object can be restored by the formulas:
$a = \operatorname{Cor} a \cup^{\mathfrak{A}} \operatorname{Edg} a$ and $a = \operatorname{Cor}' a \cup^{\mathfrak{A}} \operatorname{Edg}' a$.

13.1. Core part and atomic elements

Proposition 34 *Let* $(\mathfrak{A}; \mathfrak{Z})$ *be a filtrator with join-closed core and* $\mathfrak{Z}$ *is an atomistic lattice. Then for every* $a \in \mathfrak{A}$ *such that* $\operatorname{Cor}' a$ *exists we have*

$$\operatorname{Cor}' a = \bigcup^{\mathfrak{Z}} \{ x \mid x \text{ is an atom of } \mathfrak{Z}, x \subseteq a \} .$$

Proof

$$\begin{aligned} \operatorname{Cor}' a &= \\ \bigcup^{\mathfrak{Z}} \{ A \in \mathfrak{Z} \mid A \subseteq a \} &= \\ \bigcup^{\mathfrak{Z}} \left\{ \bigcup^{\mathfrak{Z}} \operatorname{atoms}^{\mathfrak{Z}} A \mid A \in \mathfrak{Z}, A \subseteq a \right\} &= \\ \bigcup^{\mathfrak{Z}} \bigcup \{ \operatorname{atoms}^{\mathfrak{Z}} A \mid A \in \mathfrak{Z}, A \subseteq a \} &= \\ \bigcup^{\mathfrak{Z}} \{ x \mid x \text{ is an atom of } \mathfrak{Z}, x \subseteq a \} . & \end{aligned}$$

□

14. Distributivity of core part over lattice operations

Theorem 64 *If* $(\mathfrak{A}; \mathfrak{Z})$ *is a join-closed filtrator and* $\mathfrak{A}$ *is a meet-semilattice and* $\mathfrak{Z}$ *is a complete lattice, then*

$$\operatorname{Cor}'(a \cap^{\mathfrak{A}} b) = \operatorname{Cor}' a \cap^{\mathfrak{Z}} \operatorname{Cor}' b.$$

Proof From theorem conditions follows that $\operatorname{Cor}'(a \cap^{\mathfrak{A}} b)$ exists.

We have $\operatorname{Cor}' p \subseteq p$ for every $p \in \mathfrak{A}$ because our filtrator is with join-closed core.

Obviously $\operatorname{Cor}'(a \cap^{\mathfrak{A}} b) \subseteq \operatorname{Cor}' a$ and $\operatorname{Cor}'(a \cap^{\mathfrak{A}} b) \subseteq \operatorname{Cor}' b$.

If $x \subseteq \operatorname{Cor}' a$ and $x \subseteq \operatorname{Cor}' b$ for some $x \in \mathfrak{Z}$ then $x \subseteq a$ and $x \subseteq b$, thus $x \subseteq a \cap^{\mathfrak{A}} b$ and $x \subseteq \operatorname{Cor}'(a \cap^{\mathfrak{A}} b)$. □

Theorem 65 *Let* $(\mathfrak{A};\mathfrak{Z})$ *be a semifiltered down-aligned filtrator with finitely meet-closed core* $\mathfrak{Z}$ *which is a complete atomistic lattice and* $\mathfrak{A}$ *is a distributive lattice, then* $\operatorname{Cor}'(a \cup^{\mathfrak{A}} b) = \operatorname{Cor}' a \cup^{\mathfrak{Z}} \operatorname{Cor}' b$ *for every* $a, b \in \mathfrak{A}$.

Proof $\operatorname{Cor}'(a \cup^{\mathfrak{A}} b) = \bigcup^{\mathfrak{Z}} \left\{ x \mid x \text{ is an atom of } \mathfrak{Z}, x \subseteq a \cup^{\mathfrak{A}} b \right\}$ (used proposition 34).

By the theorem 50 we have $\operatorname{Cor}'(a \cup^{\mathfrak{A}} b) = \bigcup^{\mathfrak{Z}}(\operatorname{atoms}^{\mathfrak{A}}(a \cup^{\mathfrak{A}} b) \cap \mathfrak{Z}) = \bigcup^{\mathfrak{Z}}((\operatorname{atoms}^{\mathfrak{A}} a \cup \operatorname{atoms}^{\mathfrak{A}} b) \cap \mathfrak{Z}) = \bigcup^{\mathfrak{Z}}((\operatorname{atoms}^{\mathfrak{A}} a \cap \mathfrak{Z}) \cup (\operatorname{atoms}^{\mathfrak{A}} b \cap \mathfrak{Z})) = \bigcup^{\mathfrak{Z}}(\operatorname{atoms}^{\mathfrak{A}} a \cap \mathfrak{Z}) \cup^{\mathfrak{Z}} \bigcup^{\mathfrak{Z}}(\operatorname{atoms}^{\mathfrak{A}} b \cap \mathfrak{Z})$ (used the theorem 1). Again using theorem 50, we get $\operatorname{Cor}'(a \cup^{\mathfrak{A}} b) = \bigcup^{\mathfrak{Z}} \{ x \mid x \text{ is an atom of } \mathfrak{Z}, x \subseteq a\} \cup^{\mathfrak{Z}} \bigcup^{\mathfrak{Z}} \{ x \mid x \text{ is an atom of } \mathfrak{Z}, x \subseteq b\} = \operatorname{Cor}' a \cup^{\mathfrak{Z}} \operatorname{Cor}' b$ (again used proposition 34). □

Theorem 66 *Let* $(\mathfrak{F};\mathfrak{A})$ *be a primary filtrator over a complete boolean lattice. Then* $(a \cap^{\mathfrak{F}} b)^+ = a^+ \cup^{\mathfrak{A}} b^+$ *for every* $a, b \in \mathfrak{F}$.

Proof $(\mathfrak{F};\mathfrak{A})$ is a filtered up-aligned complete lattice filtrator with finitely join-closed (theorem 23) co-separable core (theorem 38) which is a complete boolean lattice. Thus by the theorem 60

$$(a \cap^{\mathfrak{F}} b)^+ = \overline{\operatorname{Cor}(a \cap^{\mathfrak{F}} b)} = \overline{\operatorname{Cor} a \cap^{\mathfrak{A}} \operatorname{Cor} b} = \overline{\operatorname{Cor} a} \cup^{\mathfrak{A}} \overline{\operatorname{Cor} b} = a^+ \cup^{\mathfrak{A}} b^+.$$

□

Theorem 67 *Let* $(\mathfrak{A};\mathfrak{Z})$ *be a filtered distributive down-aligned, complete lattice filtrator with finitely meet-closed, separable core which is a complete atomistic boolean lattice. Then* $(a \cup^{\mathfrak{A}} b)^* = a^* \cap^{\mathfrak{Z}} b^*$ *for every* $a, b \in \mathfrak{A}$.

Proof $(a \cup^{\mathfrak{A}} b)^* = \overline{\operatorname{Cor}'(a \cup^{\mathfrak{A}} b)} = \overline{\operatorname{Cor}' a \cup^{\mathfrak{Z}} \operatorname{Cor}' b} = \overline{\operatorname{Cor}' a} \cap^{\mathfrak{Z}} \overline{\operatorname{Cor}' b} = a^* \cap^{\mathfrak{Z}} b^*$ (used the theorem 61). □

Theorem 68 *Let* $\mathfrak{A}$ *be a complete boolean lattice. Then* $(a \cap^{\mathfrak{F}} b)^* = a^* \cup^{\mathfrak{A}} b^*$ *for every* $a, b \in \mathfrak{F}$.

Proof $(\mathfrak{F};\mathfrak{A})$ is a filtered complete lattice filtrator with down-aligned, up-aligned, finitely meet-closed, separable core which is a complete boolean lattice. So

$$(a \cap^{\mathfrak{F}} b)^* = \overline{\operatorname{Cor}(a \cap^{\mathfrak{F}} b)} = \overline{\operatorname{Cor} a \cap^{\mathfrak{A}} \operatorname{Cor} b} = \overline{\operatorname{Cor} a} \cup^{\mathfrak{A}} \overline{\operatorname{Cor} b} = a^* \cup^{\mathfrak{A}} b^*$$

(used the theorem 61). □

15. Fréchet filter

The consideration below is about filters on a set U, but this can be generalized for filters on complete atomic boolean algebras due complete atomic boolean algebras are isomorpic to algebras of sets on some set U.

Definition 71 *$\{U \setminus X \mid X$ is a finite subset of $U\}$ is called either* ***Fréchet filter*** *or* ***cofinite filter****.*

It is trivial that Fréchet filter is a filter.

Definition 72 *I will call* ***Fréchet f. o.*** *and denote Ω the filter object corresponding to the Fréchet filter.*

Proposition 35 $\operatorname{Cor}\Omega = \emptyset$.

Proof This can be deduced from the formula $\forall \alpha \in U \exists X \in \operatorname{up}\Omega : \alpha \notin X$. $\Box$

Theorem 69 $\max\{\mathcal{X} \in \mathfrak{F} \mid \operatorname{Cor}\mathcal{X} = \emptyset\} = \Omega$.

Proof Due the last proposition, enough to show that $\operatorname{Cor}\mathcal{X} = \emptyset \Rightarrow \mathcal{X} \subseteq \Omega$ for every f.o. $\mathcal{X}$.

Let $\operatorname{Cor}\mathcal{X} = \emptyset$ for some f.o. $\mathcal{X}$. Let $X \in \operatorname{up}\Omega$. We need to prove that $X \in \operatorname{up}\mathcal{X}$.

$X = U \setminus \{\alpha_0, \ldots, \alpha_n\}$. $U \setminus \{\alpha_i\} \in \operatorname{up}\mathcal{X}$ because otherwise $\alpha_i \in \operatorname{Cor}\mathcal{X}$. So $X \in \operatorname{up}\mathcal{X}$. $\Box$

Theorem 70 $\Omega = \bigcup^{\mathfrak{F}}\{x \mid x$ *is a non-trivial atomic f.o.*$\}$.

Proof It follows from the facts that $\operatorname{Cor} x = \emptyset$ for every non-trivial atomic f.o. x, that $\mathfrak{F}$ is an atomistic lattice, and the previous theorem. $\Box$

Theorem 71 Cor *is the lower adjoint of* $\Omega \cup^{\mathfrak{F}} -$.

Proof Because both Cor and $\Omega \cup^{\mathfrak{F}} -$ are monotone, it is enough (theorem 8) to prove (for every filter objects $\mathcal{X}$ and $\mathcal{Y}$)

$$\mathcal{X} \subseteq \Omega \cup^{\mathfrak{F}} \operatorname{Cor}\mathcal{X} \quad \text{and} \quad \operatorname{Cor}(\Omega \cup^{\mathfrak{F}} \mathcal{Y}) \subseteq \mathcal{Y}.$$

$\operatorname{Cor}(\Omega \cup^{\mathfrak{F}} \mathcal{Y}) = \operatorname{Cor}\Omega \cup \operatorname{Cor}\mathcal{Y} = \emptyset \cup \operatorname{Cor}\mathcal{Y} = \operatorname{Cor}\mathcal{Y} \subseteq \mathcal{Y}$.

$\Omega \cup^{\mathfrak{F}} \operatorname{Cor}\mathcal{X} \supseteq \operatorname{Edg}\mathcal{X} \cup^{\mathfrak{F}} \operatorname{Cor}\mathcal{X} = \mathcal{X}$. $\Box$

Corollary 20 $\operatorname{Cor}\mathcal{X} = \mathcal{X} \setminus^{*} \Omega$ *for any f.o. on a set.*

Proof By the theorem 13. $\Box$

Corollary 21 $\operatorname{Cor}\bigcup^{\mathfrak{F}} S = \bigcup \langle\operatorname{Cor}\rangle S$ *for any set S of f.o. on a set.*

Proof By the theorem 12. $\Box$

16. Complementive filter objects and factoring by a filter

Definition 73 *Let $\mathfrak{A}$ be a $\cap$-semilattice and $\mathcal{A} \in \mathfrak{A}$. Then the relation $\sim$ on $\mathfrak{A}$ is defined by the formula*

$$\forall X, Y \in \mathfrak{A} : (X \sim Y \Leftrightarrow X \cap^{\mathfrak{A}} \mathcal{A} = Y \cap^{\mathfrak{A}} \mathcal{A}).$$

Proposition 36 *The relation $\sim$ is an equivalence relation.*

Proof

Reflexivity Obvious.

Symmetry Obvious

Transitivity Obvious.

□

Proposition 37 *Let $\mathfrak{A}$ be a distribitive lattice, $\mathcal{A} \in \mathfrak{F}$. Then for every $X, Y \in \mathfrak{A}$*

$$X \sim Y \Leftrightarrow \exists A \in \operatorname{up} \mathcal{A} : X \cap^{\mathfrak{A}} A = Y \cap^{\mathfrak{A}} A.$$

Proof $\exists A \in \operatorname{up} \mathcal{A} : X \cap^{\mathfrak{A}} A = Y \cap^{\mathfrak{A}} A \Leftrightarrow \exists A \in \operatorname{up} \mathcal{A} : X \cap^{\mathfrak{F}} A = Y \cap^{\mathfrak{F}} A \Rightarrow \exists A \in \operatorname{up} \mathcal{A} : X \cap^{\mathfrak{F}} A \cap^{\mathfrak{F}} \mathcal{A} = Y \cap^{\mathfrak{F}} A \cap^{\mathfrak{F}} \mathcal{A} \Leftrightarrow \exists A \in \operatorname{up} \mathcal{A} : X \cap^{\mathfrak{F}} \mathcal{A} = Y \cap^{\mathfrak{F}} \mathcal{A} \Leftrightarrow X \cap^{\mathfrak{F}} \mathcal{A} = Y \cap^{\mathfrak{F}} \mathcal{A} \Leftrightarrow X \sim Y$.

On the other hand, $X \cap^{\mathfrak{F}} \mathcal{A} = Y \cap^{\mathfrak{F}} \mathcal{A} \Leftrightarrow \left\{ X \cap^{\mathfrak{A}} A_0 \mid A_0 \in \operatorname{up} \mathcal{A} \right\} = \left\{ Y \cap^{\mathfrak{A}} A_1 \mid A_1 \in \operatorname{up} \mathcal{A} \right\} \Rightarrow \exists A_0, A_1 \in \operatorname{up} \mathcal{A} : X \cap^{\mathfrak{A}} A_0 = Y \cap^{\mathfrak{A}} A_1 \Rightarrow \exists A_0, A_1 \in \operatorname{up} \mathcal{A} : X \cap^{\mathfrak{A}} A_0 \cap^{\mathfrak{A}} A_1 = Y \cap^{\mathfrak{A}} A_0 \cap^{\mathfrak{A}} A_1 \Rightarrow \exists A \in \operatorname{up} \mathcal{A} : X \cap^{\mathfrak{A}} A = Y \cap^{\mathfrak{A}} A$. □

Proposition 38 *The relation $\sim$ is a congruence for each of the following:*

1. *a $\cap$-semilattice $\mathfrak{A}$;*
2. *a distribitive lattice $\mathfrak{A}$.*

Proof Let $a_0, a_1, b_0, b_1 \in \mathfrak{A}$ and $a_0 \sim a_1$ and $b_0 \sim b_1$. □

1. $a_0 \cap b_0 \sim a_1 \cap b_1$ because

$$(a_0 \cap b_0) \cap \mathcal{A} = a_0 \cap (b_0 \cap \mathcal{A}) = a_0 \cap (b_1 \cap \mathcal{A}) = b_1 \cap (a_0 \cap \mathcal{A}) = b_1 \cap (a_1 \cap \mathcal{A}) = (a_1 \cap b_1) \cap \mathcal{A}.$$

2. Taking the above into account, we need to prove only $a_0 \cup b_0 \sim a_1 \cup b_1$. We have

$$(a_0 \cup b_0) \cap \mathcal{A} = (a_0 \cap \mathcal{A}) \cup (b_0 \cap \mathcal{A}) = (a_1 \cap \mathcal{A}) \cup (b_1 \cap \mathcal{A}) = (a_1 \cup b_1) \cap \mathcal{A}.$$

Definition 74 *We will denote $A/(\sim) = A/((\sim) \cap A \times A)$ for a set A and an equivalence relation $\sim$ on a set $B \supseteq A$. I will call $\sim$ a congruence on A when $(\sim) \cap A \times A$ is a congruence on A.*

Theorem 72 *Let $\mathfrak{F}$ be the set of filters over a boolean lattice $\mathfrak{A}$ and $\mathcal{A} \in \mathfrak{F}$. Consider the function $\gamma : Z(D\mathcal{A}) \to \mathfrak{A}/\sim$ defined by the formula (for every $p \in Z(D\mathcal{A})$)*

$$\gamma p = \left\{ X \in \mathfrak{A} \mid X \cap^{\mathfrak{F}} \mathcal{A} = p \right\}.$$

Then:

1. *γ is a lattice isomorphism.*

2. *$\forall Q \in q : \gamma^{-1} q = Q \cap^{\mathfrak{F}} \mathcal{A}$ for every $q \in \mathfrak{A}/\sim$.*

Proof $\forall p \in Z(D\mathcal{A}) : \gamma p \neq \emptyset$ because of the theorem 59. Thus easy to see that $\gamma p \in \mathfrak{A}/\sim$ and that γ is an injection.

Let's prove that γ is a lattice homomorphism:

$\gamma(p_0 \cap^{\mathfrak{F}} p_1) = \left\{ X \in \mathfrak{A} \mid X \cap^{\mathfrak{F}} \mathcal{A} = p_0 \cap^{\mathfrak{F}} p_1 \right\}$;

$$\begin{aligned} \gamma p_0 \cap^{\mathfrak{A}/\sim} \gamma p_1 &= \\ \left\{ X_0 \in \mathfrak{A} \mid X_0 \cap^{\mathfrak{F}} \mathcal{A} = p_0 \right\} \cap^{\mathfrak{A}/\sim} \left\{ X_1 \in \mathfrak{A} \mid X_1 \cap^{\mathfrak{F}} \mathcal{A} = p_1 \right\} &= \\ \left\{ X_0 \cap^{\mathfrak{F}} X_1 \mid X_0, X_1 \in \mathfrak{A}, X_0 \cap^{\mathfrak{F}} \mathcal{A} = p_0 \wedge X_1 \cap^{\mathfrak{F}} \mathcal{A} = p_1 \right\} &\subseteq \\ \left\{ X' \in \mathfrak{A} \mid X' \cap^{\mathfrak{F}} \mathcal{A} = p_0 \cap^{\mathfrak{F}} p_1 \right\} &= \\ \gamma(p_0 \cap^{\mathfrak{F}} p_1). \end{aligned}$$

Because $\gamma p_0 \cap^{\mathfrak{A}/\sim} \gamma p_1$ and $\gamma(p_0 \cap^{\mathfrak{F}} p_1)$ are equivalence classes, thus follows $\gamma p_0 \cap^{\mathfrak{A}/\sim} \gamma p_1 = \gamma(p_0 \cap^{\mathfrak{F}} p_1)$.

$\gamma(p_0 \cup^{\mathfrak{F}} p_1) = \left\{ X \in \mathfrak{A} \mid X \cap^{\mathfrak{F}} \mathcal{A} = p_0 \cup^{\mathfrak{F}} p_1 \right\}$;

$$\begin{aligned} \gamma p_0 \cup^{\mathfrak{A}/\sim} \gamma p_1 &= \\ \left\{ X_0 \in \mathfrak{A} \mid X_0 \cap^{\mathfrak{F}} \mathcal{A} = p_0 \right\} \cup^{\mathfrak{A}/\sim} \left\{ X_1 \in \mathfrak{A} \mid X_1 \cap^{\mathfrak{F}} \mathcal{A} = p_1 \right\} &= \\ \left\{ X_0 \cup^{\mathfrak{F}} X_1 \mid X_0, X_1 \in \mathfrak{A}, X_0 \cap^{\mathfrak{F}} \mathcal{A} = p_0 \wedge X_1 \cap^{\mathfrak{F}} \mathcal{A} = p_1 \right\} &\subseteq \\ \left\{ X_0 \cup^{\mathfrak{F}} X_1 \mid X_0, X_1 \in \mathfrak{A}, (X_0 \cup^{\mathfrak{F}} X_1) \cap^{\mathfrak{F}} \mathcal{A} = p_0 \cup^{\mathfrak{F}} p_1 \right\} &\subseteq \\ \left\{ X' \in \mathfrak{A} \mid X' \cap^{\mathfrak{F}} \mathcal{A} = p_0 \cup^{\mathfrak{F}} p_1 \right\} &= \\ \gamma(p_0 \cup^{\mathfrak{F}} p_1). \end{aligned}$$

Because $\gamma p_0 \cup^{\mathfrak{A}/\sim} \gamma p_1$ and $\gamma(p_0 \cup^{\mathfrak{F}} p_1)$ are equivalence classes, thus follows $\gamma p_0 \cup^{\mathfrak{A}/\sim} \gamma p_1 = \gamma(p_0 \cup^{\mathfrak{F}} p_1)$.

To finish the proof it's enough to show that $\forall Q \in q : q = \gamma(Q \cap^{\mathfrak{F}} \mathcal{A})$ for every $q \in \mathfrak{A}/\sim$. (From this follows that γ is surjective because q is not empty and thus $\exists Q \in q : q = \gamma(Q \cap^{\mathfrak{F}} \mathcal{A})$.) Really,

$$\gamma(Q \cap^{\mathfrak{F}} \mathcal{A}) = \left\{ X \in \mathfrak{A} \mid X \cap^{\mathfrak{F}} \mathcal{A} = Q \cap^{\mathfrak{F}} \mathcal{A} \right\} = [Q] = q.$$

$\square$

This isomorphism is useful in both directions to reveal properties of both lattices $Z(D\mathcal{A})$ and $(\mathcal{P}U)/\sim$.

Corollary 22 *If $\mathfrak{A}$ is a boolean lattice then $\mathfrak{A}/\sim$ is a boolean lattice.*

Proof Because $Z(D\mathcal{A})$ is a boolean lattice (theorem 6). $\square$

17. Number of filters on a set

Theorem 73 *Let U be a set. The number of atomic f.o. on U is $2^{2^{\operatorname{card} U}}$ if U is infinite and $\operatorname{card} U$ if U is finite.*

Proof See [10]. $\square$

Corollary 23 *The number of filters on U is $2^{2^{\operatorname{card} U}}$ if U is infinite and $2^{\operatorname{card} U}$ if U is finite.*

Proof The finite case is obvious. The infinite case follows from the theorem and the fact that filters are collections of sets and there cannot be more than $2^{2^{\operatorname{card} U}}$ collections of sets on U. $\square$

18. Partitioning filter objects

Definition 75 *Let $\mathfrak{A}$ be a complete lattice. **Thorning** of an element $a \in \mathfrak{A}$ is a set $S \in \mathcal{P}\mathfrak{A} \setminus \{0\}$ such that*

$$\bigcup^{\mathfrak{A}} S = a \quad \text{and} \quad \forall x, y \in S : x \asymp^{\mathfrak{A}} y.$$

Definition 76 *Let $\mathfrak{A}$ be a complete lattice. **Weak partition** of an element $a \in \mathfrak{A}$ is a set $S \in \mathcal{P}\mathfrak{A} \setminus \{0\}$ such that*

$$\bigcup^{\mathfrak{A}} S = a \quad \text{and} \quad \forall x \in S : x \asymp^{\mathfrak{A}} \bigcup^{\mathfrak{A}} (S \setminus \{x\}).$$

Definition 77 *Let $\mathfrak{A}$ be a complete lattice. **Strong partition** of an element $a \in \mathfrak{A}$ is a set $S \in \mathcal{P}\mathfrak{A} \setminus \{0\}$ such that*

$$\bigcup^{\mathfrak{A}} S = a \quad \text{and} \quad \forall A, B \in \mathcal{P}S : \left(A \asymp B \Rightarrow \bigcup^{\mathfrak{A}} A \asymp^{\mathfrak{A}} \bigcup^{\mathfrak{A}} B \right).$$

Obvious 23 *1. Every strong partition is a weak partition.*

2. Every weak partition is a thorning.

See the section "Open problems" for supposed properties of partitions.

19. Open problems

In this section I will formulate some conjectures about lattices of filter objects on a set. If a conjecture comes true, it may be generalized for more general lattices (such as, for example, lattices of filters on arbitrary lattices). I deem that the main challenge is to prove the special case about lattices of filter objects on a set, and generalizing the conjectures is expected to be a simple task.

19.1. Partitioning

Consider the complete lattice $[S]$ generated by the set S where S is a strong partition of some element a.

Conjecture 1 $[S] = \left\{ \bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S \right\}$*, where* $[S]$ *is the complete lattice generated by a strong partition* S *of an element of a lattice* $\mathfrak{F}$ *of filter objects on a set.*

Proposition 39 *Provided that the last conjecture is true, we have that* $[S]$ *is a complete atomic boolean lattice with the set of its atoms being* S*.*

Remark 13 Consequently S is atomistic, completely distributive and isomorphic to a power set algebra (see [13]).

Proof Completeness of $[S]$ is obvious. Let $A \in [S]$. Then exists $X \in \mathcal{P}S$ such that $A = \bigcup^{\mathfrak{F}} X$. Let $B = \bigcup^{\mathfrak{F}} (S \setminus X)$. Then $B \in [S]$ and $A \cap B = 0$. $A \cup B = \bigcup^{\mathfrak{F}} S$ is the biggest element of $[S]$. So we have proved that $[S]$ is a boolean lattice.

Now let prove that $[S]$ is atomic with the set of atoms being S. Let $z \in S$ and $A \in [S]$. If $A \neq z$ then either $A = 0$ or $x \in X$ where $A = \bigcup^{\mathfrak{F}} X$, $X \in \mathcal{P}S$ and $x \neq z$. Because S is a partition, $\bigcup^{\mathfrak{F}} (X \setminus \{z\}) \cap^{\mathfrak{F}} z = 0$ and $\bigcup^{\mathfrak{F}} (X \setminus \{z\}) \neq 0$. So $A = \bigcup^{\mathfrak{F}} X = \bigcup^{\mathfrak{F}} (X \setminus \{z\}) \cup^{\mathfrak{F}} z \nsubseteq z$.

Finally we will prove that elements of $[S] \setminus S$ are not atoms. Let $A \in [S] \setminus S$ and $A \neq 0$. Then $A \supseteq x \cup^{\mathfrak{F}} y$ where $x, y \in S$ and $x \neq y$. If A is an atom then $A = x = y$ what is impossible. □

Proposition 40 *The conjecture about the value of* $[S]$ *is equivalent to closedness of* $\left\{ \bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S \right\}$ *under arbitrary meets and joins.*

Proof If $\left\{ \bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S \right\} = [S]$ then trivially $\left\{ \bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S \right\}$ is closed under arbitrary meets and joins.

If $\left\{ \bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S \right\}$ is closed under arbitrary meets and joins, then it is the complete lattice generated by the set S because it cannot be smaller than the set of all suprema of subsets of S. □

That $\left\{ \bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S \right\}$ is closed under arbitrary joins is trivial. I have not succeeded to prove that it is closed under arbitrary meets, but have proved a weaker statement that is is closed under finite meets:

Proposition 41 $\left\{\bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S\right\}$ *is closed under finite meets.*

Proof Let $R = \left\{\bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S\right\}$. Then

$$\begin{aligned}
\bigcup{}^{\mathfrak{A}} X \cap^{\mathfrak{F}} \bigcup{}^{\mathfrak{A}} Y &= \\
\bigcup{}^{\mathfrak{A}}((X \cap Y) \cup (X \setminus Y)) \cap^{\mathfrak{F}} \bigcup{}^{\mathfrak{A}} Y &= \\
\left(\bigcup{}^{\mathfrak{A}}(X \cap Y) \cup^{\mathfrak{F}} \bigcup{}^{\mathfrak{A}}(X \setminus Y)\right) \cap^{\mathfrak{F}} \bigcup{}^{\mathfrak{A}} Y &= \\
\left(\bigcup{}^{\mathfrak{A}}(X \cap Y) \cap^{\mathfrak{F}} \bigcup{}^{\mathfrak{A}} Y\right) \cup^{\mathfrak{F}} \left(\bigcup{}^{\mathfrak{A}}(X \setminus Y) \cap^{\mathfrak{F}} \bigcup{}^{\mathfrak{A}} Y\right) &= \\
\left(\bigcup{}^{\mathfrak{A}}(X \cap Y) \cap^{\mathfrak{F}} \bigcup{}^{\mathfrak{A}} Y\right) \cup^{\mathfrak{F}} 0 &= \\
\bigcup{}^{\mathfrak{A}}(X \cap Y) \cap^{\mathfrak{F}} \bigcup{}^{\mathfrak{A}} Y.&
\end{aligned}$$

Applying the formula $\bigcup^{\mathfrak{A}} X \cap^{\mathfrak{F}} \bigcup^{\mathfrak{A}} Y = \bigcup^{\mathfrak{A}}(X \cap Y) \cap^{\mathfrak{F}} \bigcup^{\mathfrak{A}} Y$ twice we get

$$\begin{aligned}
\bigcup{}^{\mathfrak{A}} X \cap^{\mathfrak{F}} \bigcup{}^{\mathfrak{A}} Y &= \\
\bigcup{}^{\mathfrak{A}}(X \cap Y) \cap^{\mathfrak{F}} \bigcup{}^{\mathfrak{A}}(Y \cap (X \cap Y)) &= \\
\bigcup{}^{\mathfrak{A}}(X \cap Y) \cap^{\mathfrak{F}} \bigcup{}^{\mathfrak{A}}(X \cap Y) &= \\
\bigcup{}^{\mathfrak{A}}(X \cap Y).&
\end{aligned}$$

But for any $A, B \in R$ exist $X, Y \in \mathcal{P}S$ such that $A = \bigcup^{\mathfrak{A}} X$ and $B = \bigcup^{\mathfrak{A}} Y$. So $A \cap^{\mathfrak{F}} B = \bigcup^{\mathfrak{A}} X \cap^{\mathfrak{F}} \bigcup^{\mathfrak{A}} Y = \bigcup^{\mathfrak{A}}(X \cap Y) \in R$. □

Conjecture 2

1. *Every filter object on a set can be partitioned into atomic filter objects.*
2. *This partition is unique.*

19.2. Quasidifference

Problem 1 Which of the following expressions are pairwise equal for all $a, b \in \mathfrak{F}$ for each lattice $\mathfrak{F}$ of filters on a set? (If some are not equal, provide counter-examples.)

1. $\bigcap^{\mathfrak{F}} \left\{z \in \mathfrak{F} \mid a \subseteq b \cup^{\mathfrak{F}} z\right\}$ (quasidifference of a and b);
2. $\bigcup^{\mathfrak{F}} \left\{z \in \mathfrak{F} \mid z \subseteq a \wedge z \cap^{\mathfrak{F}} b = \emptyset\right\}$ (second quasidifference of a and b);
3. $\bigcup^{\mathfrak{F}}(\operatorname{atoms}^{\mathfrak{F}} a \setminus \operatorname{atoms}^{\mathfrak{F}} b)$;
4. $\bigcup^{\mathfrak{F}} \left\{a \cap^{\mathfrak{F}} (U \setminus B) \mid B \in \operatorname{up} b\right\}$.

19.3. Non-formal problems

Should we introduce the concept of star objects, analogous to filter objects, and research the lattice of star objects?

Find a common generalization of two theorems:

1. If $\mathfrak{A}$ is a meet-semilattice with greatest element 1 then for any $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$

$$\mathrm{up}(\mathcal{A} \cup^{\mathfrak{F}} \mathcal{B}) = \mathrm{up}\,\mathcal{A} \cap \mathrm{up}\,\mathcal{B}.$$

2. If $\mathfrak{A}$ is a join-semilattice then $\mathfrak{F}$ is a join-semilattice then and for any $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$

$$\mathrm{up}(\mathcal{A} \cup^{\mathfrak{F}} \mathcal{B}) = \mathrm{up}\,\mathcal{A} \cap \mathrm{up}\,\mathcal{B}.$$

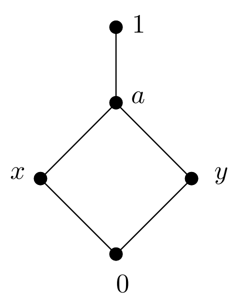
Under which conditions $a \setminus^* b$ and $a \# b$ are complementive to a?

Generalize straight maps for arbitrary posets.

Appendix A. Some counter-examples

Example 1 There exist a bounded distributive lattice which is not lattice with separable center.

Proof The lattice with the following Hasse diagram is bounded and distributive because it does not contain “diamond lattice” nor “pentagon lattice” as a sublattice [14].



It's center is $\{0, 1\}$. $x \cap y = 0$ indeed $\mathrm{up}\, x = \{1\}$ but $1 \cap y \neq 0$ consequently the lattice is not with separable center. □

For further examples we will use the filter object Δ defined by the formula

$$\Delta = \bigcap\nolimits^{\mathfrak{F}} \{(-\varepsilon; \varepsilon) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0\}$$

and more general

$$\Delta + a = \bigcap\nolimits^{\mathfrak{F}} \{(a - \varepsilon; a + \varepsilon) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0\}.$$

Example 2 There exists $A \in \mathcal{P}U$ such that $\bigcap^{\mathfrak{F}} A \neq \bigcap^{\mathcal{P}U} A$ for some set U.

Proof $\bigcap^{\mathcal{P}\mathbb{R}} \{(-a; a) \mid a \in \mathbb{R}, a > 0\} = \{0\} \neq \Delta$. $\square$

Example 3 There exists a set U and there are a f.o. a and a set S of f.o. on the lattice $\mathcal{P}U$ such that $a \cap^{\mathfrak{F}} \bigcup^{\mathfrak{F}} S \neq \bigcup^{\mathfrak{F}} \langle a \cap^{\mathfrak{F}} \rangle S$.

Proof Let $a = \Delta$ and $S = \{(\varepsilon; +\infty) \mid \varepsilon > 0\}$. Then $a \cap^{\mathfrak{F}} \bigcup^{\mathfrak{F}} S = \Delta \cap^{\mathfrak{F}} (0; +\infty) \neq \emptyset$ while $\bigcup^{\mathfrak{F}} \langle a \cap^{\mathfrak{F}} \rangle S = \bigcup^{\mathfrak{F}} \{\emptyset\} = \emptyset$. $\square$

Example 4 There are thornings which are not weak partitions.

Proof $\{\Delta + a \mid a \in \mathbb{R}\}$ is a thorning but not weak partition of the real line. $\square$

Lemma 7 *Let $\mathfrak{F}$ be the set of f.o. on a set U. Then $X \cap^{\mathfrak{F}} \Omega \subseteq Y \cap^{\mathfrak{F}} \Omega$ iff $X \setminus Y$ is a finite set, for every sets $X, Y \in \mathcal{P}U$.*

Proof $X \cap^{\mathfrak{F}} \Omega \subseteq Y \cap^{\mathfrak{F}} \Omega \Leftrightarrow \{X \cap K_X \mid K_X \in \operatorname{up} \Omega\} \supseteq \{Y \cap K_Y \mid K_Y \in \operatorname{up} \Omega\} \Leftrightarrow \forall K_Y \in \operatorname{up} \Omega \exists K_X \in \operatorname{up} \Omega : Y \cap K_Y = X \cap K_X \Leftrightarrow \forall L_Y \in M \exists L_X \in M : Y \setminus L_Y = X \setminus L_X \Leftrightarrow \forall L_Y \in M : X \setminus (Y \setminus L_Y) \in M \Leftrightarrow X \setminus Y \in M$ where M is the set of finite subsets of U. $\square$

Example 5 There exists a filter object $\mathcal{A}$ on a set U such that $(\mathcal{P}U)/\sim$ and $Z(D\mathcal{A})$ are not complete lattices.

Proof Due isomorphism it's enough to prove for $(\mathcal{P}U)/\sim$.

Let's take $U = \mathbb{N}$ and $\mathcal{A} = \Omega$ be the Frechet filter object on $\mathbb{N}$.

Partition $\mathbb{N}$ into infinitely many infinite sets $A_0, A_1, \ldots$. To withhold our example we will prove that the set $\{[A_0], [A_1], \ldots\}$ has no supremum in $(\mathcal{P}U)/\sim$.

Let $[X]$ be an upper bound of $[A_0], [A_1], \ldots$ that is $\forall i \in \mathbb{N} : X \cap \Omega \supseteq A_i \cap \Omega$ that is $A_i \setminus X$ is finite. Consequently X is infinite. So $X \cap A_i \neq \emptyset$.

Choose for every $i \in \mathbb{N}$ some $z_i \in X \cap A_i$. Then $\{z_0, z_1, \ldots\}$ is an infinite subset of X (take in account that $z_i \neq z_j$ for $i \neq j$). Let $Y = X \setminus \{z_0, z_1, \ldots\}$. Then $Y \cap^{\mathfrak{F}} \Omega \supseteq A_i \cap^{\mathfrak{F}} \Omega$ because $A_i \setminus Y = A_i \setminus (X \setminus \{z_i\}) = (A_i \setminus X) \cup \{z_i\}$ which is finite because $A_i \setminus X$ is finite. Thus $[Y]$ is an upper bound for $\{[A_0], [A_1], \ldots\}$.

Suppose $Y \cap^{\mathfrak{F}} \Omega = X \cap^{\mathfrak{F}} \Omega$. Then $Y \setminus X$ is finite what is not true. So $Y \cap^{\mathfrak{F}} \Omega \subset X \cap^{\mathfrak{F}} \Omega$ that is $[Y]$ is below $[X]$. $\square$

Appendix A.1. Weak and strong partition

Definition 78 *A family S of subsets of a countable set is **independent** iff the intersection of any finitely many members of S and the complements of any other finitely many members of S is infinite.*

Lemma 8 *The "infinite" at the end of the definition could be equivalently replaced with "nonempty" if we assume that S is infinite.*

Proof Suppose that some sets from the above definition has a finite intersection J of cardinality n. Then (thanks S is infinite) get one more set $X \in S$ and we have $J \cap X \neq \emptyset$ and $J \cap (\mathbb{N} \setminus X) \neq \emptyset$. So $\operatorname{card}(J \cap X) < n$. Repeating this, we prove that for some finite family of sets we have empty intersection what is a contradiction. $\square$

Lemma 9 *There exists an independent family on $\mathbb{N}$ of cardinality $\mathfrak{c}$.*

Proof Let C be the set of finite subsets of $\mathbb{Q}$. Since $\operatorname{card} C = \operatorname{card} \mathbb{N}$, it suffices to find $\mathfrak{c}$ independent subsets of C. For each $r \in \mathbb{R}$ let

$$E_r = \{ F \in C \mid \operatorname{card}(F \cap (-\infty; r)) \text{ is even} \}.$$

All E_{r_1} and E_{r_2} are distinct for distinct $r_1, r_2 \in \mathbb{R}$ since we may consider $F = \{r'\} \in C$ where a rational number r' is between r_1 and r_2 and thus F is a member of exactly one of the sets E_{r_1} and E_{r_2}. Thus card $\{ E_r \mid r \in \mathbb{R} \} = \mathfrak{c}$.

We will show that $\{ E_r \mid r \in \mathbb{R} \}$ is independent. Let $r_1, \ldots, r_k, s_1, \ldots, s_k$ be distinct reals. Enough to show that these have a nonempty intersection, that is existence of some F such that F belongs to all the E_r and none of E_s.

But this can be easily accomplished taking F having zero or one element in each of intervals to which $r_1, \ldots, r_k, s_1, \ldots, s_k$ split the real line. $\square$

Example 6 There exists a weak partition which is not a strong partition.

Proof (suggested by Andreas Blass) Let $\{ X_r \mid r \in \mathbb{R} \}$ be an independent family of subsets of $\mathbb{N}$.

Let $\mathcal{F}_a$ be a filter object generated by X_a and the complements $\mathbb{N} \setminus X_b$ for all $b \in \mathbb{R}, b \neq a$. Independece implies that $\mathcal{F}_a \neq \emptyset$ (by properties of filter bases).

Let $S = \{ \mathcal{F}_r \mid r \in \mathbb{R} \}$. We will prove that S is a weak partition but not a strong partition.

Let $a \in \mathbb{R}$. Then $X_a \in \operatorname{up} \mathcal{F}_a$ while $\forall b \in \mathbb{R} \setminus \{a\} : \mathbb{N} \setminus X_a \in \operatorname{up} \mathcal{F}_b$ and therefore $\mathbb{N} \setminus X_a \in \operatorname{up} \bigcup^{\mathfrak{F}} \{ \mathcal{F}_b \mid \mathbb{R} \ni b \neq a \}$. Therefore $\mathcal{F}_a \cap^{\mathfrak{F}} \bigcup^{\mathfrak{F}} \{ \mathcal{F}_b \mid \mathbb{R} \ni b \neq a \} = \emptyset$. Thus S is a weak partition.

Suppose S is a strong partition. Then for each set $Z \in \mathcal{P}\mathbb{R}$

$$\bigcup^{\mathfrak{F}} \{ \mathcal{F}_b \mid b \in Z \} \cap^{\mathfrak{F}} \bigcup^{\mathfrak{F}} \{ \mathcal{F}_b \mid b \in \mathbb{R} \setminus Z \} = \emptyset$$

what is equivalent to existence of $M(Z) \in \mathcal{P}\mathbb{N}$ such that

$$M(Z) \in \operatorname{up} \bigcup^{\mathfrak{F}} \{ \mathcal{F}_b \mid b \in Z \} \quad \text{and} \quad \mathbb{N} \setminus M(Z) \in \operatorname{up} \bigcup^{\mathfrak{F}} \{ \mathcal{F}_b \mid b \in \mathbb{R} \setminus Z \} = \emptyset$$

that is

$$\forall b \in Z : M(Z) \in \operatorname{up} \mathcal{F}_b \quad \text{and} \quad \forall b \in \mathbb{R} \setminus Z : \mathbb{N} \setminus M(Z) \in \operatorname{up} \mathcal{F}_b.$$

Suppose $Z \neq Z' \in \mathcal{P}\mathbb{N}$. Without loss of generality we may assume that some $b \in Z$ but $b \notin Z'$. Then $M(Z) \in \operatorname{up} \mathcal{F}_b$ and $\mathbb{N} \setminus M(Z') \in \operatorname{up} \mathcal{F}_b$. If $M(Z) = M(Z')$ then $\mathcal{F}_b = \emptyset$ what contradicts to the above.

So M is an injective function from $\mathcal{P}\mathbb{R}$ to $\mathcal{P}\mathbb{N}$ what is impossible due cardinality issues. □

Appendix B. Logic of Generalizations

In mathematics it is often encountered that a smaller set S naturally bijectively corresponds to a subset R of a larger set B. (In other words, there is specified an injection from S to B.) It is a widespread practice to equate S with R.

Remark 14 I denote the first set S from the first letter of the word "small" and the second set B from the first letter of the word "big", because S is intuitively considered as smaller than B. (However we do not require $\operatorname{card} S < \operatorname{card} B$.)

The set B is considered as a generalization of the set S, for example: whole numbers generalizing natural numbers, rational numbers generalizing whole numbers, real numbers generalizing rational numbers, complex numbers generalizing real numbers, etc.

But strictly speaking this equating may contradict to the axioms of ZF/ZFC because we are not insured against $S \cap B \neq \emptyset$ incidents. Not wonderful, as it is often labeled as "without proof".

To work around of this (and formulate things exactly what could benefit computer proof assistants) we will replace the set B with a new set B' having a bijection $M : B \to B'$ such that $S \subseteq B'$. (I call this bijection M from the first letter of the word "move" which signifies the move from the old set B to a new set B').

Appendix B.1. The formalistic

Let S and B be sets. Let E be an injection from S to B. Let $R = \operatorname{im} E$.

Let $t = \mathcal{P} \bigcup\bigcup S$.

Let $M(x) = \begin{cases} E^{-1}x & \text{if } x \in R; \\ (t; x) & \text{if } x \notin R. \end{cases}$

Recall that in standard ZF $(t; x) = \{\{t\}, \{t, x\}\}$ by definition.

Theorem 74 *$(t; x) \notin S$.*

Proof Suppose $(t; x) \in S$. Then $\{\{t\}, \{t, x\}\} \in S$. Consequently $\{t\} \in \bigcup S$; $\{t\} \subseteq \bigcup\bigcup S$; $\{t\} \in \mathcal{P}\bigcup\bigcup S$; $\{t\} \in t$ what contradicts to the axiom of foundation (aka axiom of regularity). □

Definition 79 *Let $B' = \operatorname{im} M$.*

Theorem 75 $S \subseteq B'$.

Proof Let $x \in S$. Then $Ex \in R$; $M(Ex) = E^{-1}Ex = x$; $x \in \operatorname{im} M = B'$. $\square$

Obvious 24 *E is a bijection from S to R.*

Theorem 76 *M is a bijection from B to B'.*

Proof Surjectivity of M is obvious. Let's prove injectivity. Let $a, b \in B$ and $M(a) = M(b)$. Consider all cases:

$a, b \in R$ $M(a) = E^{-1}a$; $M(b) = E^{-1}b$; $E^{-1}a = E^{-1}b$; thus $a = b$ because E^{-1} is a bijection.

$a \in R$**,** $b \notin R$ $M(a) = E^{-1}a$; $M(b) = (t; b)$; $M(a) \in S$; $M(b) \notin S$. Thus $M(a) \neq M(b)$.

$a \notin R$**,** $b \in R$ Analogous,

$a, b \notin R$ $M(a) = (t; a)$; $M(b) = (t; b)$. Thus $M(a) = M(b)$ implies $a = b$.

$\square$

Theorem 77 $M \circ E = \mathrm{id}_S$.

Proof Let $x \in S$. Then $Ex \in R$; $M(Ex) = E^{-1}Ex = x$. $\square$

Obvious 25 $E = M^{-1}|_S$.

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