Funcoids are Filters

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1 Draft status

This is a rough draft.

In this article notations are used accordingly:

http://www.mathematics21.org/binaries/rewrite-plan.pdf

Particularly $\langle f\rangle^{*}X\overset{\mathrm{def}}{=}\{y\mid x\in X\wedge x\,f\,y\}$ for a binary relation f and a set X.

The motto of this article is: "Funcoids are filters on a lattice."

2 Rearrangement of collections of sets

Let Q is a set of sets.

Let $\equiv$ be the relation on $\bigcup Q$ defined by the formula

$$a\equiv b\Leftrightarrow\forall X\in Q\colon(a\in X\Leftrightarrow b\in X).$$

[TODO: Generalize it by the formula $a\equiv b\Leftrightarrow\forall X\in Q\colon(a\in\operatorname{atoms}X\Leftrightarrow b\in\operatorname{atoms}X)$.]

[TODO: Reloids $\mathsf{RLD}(\mathfrak{A};\mathfrak{B})$ between posets $\mathfrak{A}$ and $\mathfrak{B}$ is $\mathfrak{F}(\operatorname{atoms}^{\mathfrak{A}}\times\operatorname{atoms}^{\mathfrak{B}})$?]

Proposition 1. $\equiv$ is an equivalence relation on $\bigcup Q$.

Proof.

Reflexivity. Obvious.

Symmetry. Obvious.

Transitivity. Let $a\equiv b\wedge b\equiv c$. Then $a\in X\Leftrightarrow b\in X\Leftrightarrow c\in X$ for every $X\in Q$. Thus $a\equiv c$. □

Definition 2. *Rearrangement* $\mathfrak{R}(Q)$ of Q is the set of equivalence classes of $\bigcup Q$ for $\equiv$.

Obvious 3. $\bigcup\mathfrak{R}(Q)=\bigcup Q$.

Obvious 4. $\emptyset\notin\mathfrak{R}(Q)$.

Lemma 5. $\operatorname{card}\mathfrak{R}(Q)\leq 2^{\operatorname{card}Q}$.

Proof. Having an equivalence class C, we can find the set $f\in\mathscr{P}Q$ of all $X\in Q$ such that $a\in X$ for all $a\in C$. $b\equiv a\Leftrightarrow\forall X\in Q\colon(a\in X\Leftrightarrow b\in X)\Leftrightarrow\forall X\in Q\colon(X\in f\Leftrightarrow b\in X)$. So $C=\{b\in\bigcup Q\mid b\equiv a\}$ can be restored knowing f. Consequently there are no more than $\operatorname{card}\mathscr{P}Q=2^{\operatorname{card}Q}$ classes. □

Corollary 6. If Q is finite, then $\mathfrak{R}(Q)$ is finite.

Proposition 7. If $X\in Q$, $Y\in\mathfrak{R}(Q)$ then $X\cap Y\neq\emptyset\Leftrightarrow Y\subseteq X$.

Proof. Let $X\cap Y\neq\emptyset$ and $x\in X\cap Y$. Then $y\in Y\Leftrightarrow x\equiv y\Leftrightarrow\forall X'\in Q\colon(x\in X'\Leftrightarrow y\in X')\Rightarrow(x\in X\Leftrightarrow y\in X)\Leftrightarrow y\in X$ for every y. Thus $Y\subseteq X$.

$Y\subseteq X\Rightarrow X\cap Y\neq\emptyset$ because $Y\neq\emptyset$. □

Proposition 8. If $\emptyset\neq X\in Q$ then there exists $Y\in\mathfrak{R}(Q)$ such that $Y\subseteq X\wedge X\cap Y\neq\emptyset$.

Proof. Let $a\in X$. Then $[a]=\{b\in\bigcup Q\mid\forall X'\in Q\colon(a\in X'\Leftrightarrow b\in X')\}=\{b\in\bigcup Q\mid\forall X'\in Q\colon b\in X'\}\subseteq\{b\in\bigcup Q\mid b\in X\}=X$. But $[a]\in\mathfrak{R}(Q)$.

$X\cap Y\neq\emptyset$ follows from $Y\subseteq X$ by the previous proposition. □

Proposition 9. If $X \in Q$ then $X = \bigcup (\mathfrak{R}(Q) \cap \mathscr{P}X)$.

Proof. $\bigcup (\mathfrak{R}(Q) \cap \mathscr{P}X) \subseteq X$ is obvious.

Let $x \in X$. Then there is $Y \in \mathfrak{R}(Q)$ such that $x \in Y$. We have $Y \subseteq X$ that is $Y \in \mathscr{P}X$ by a proposition above. So $x \in Y$ where $Y \in \mathfrak{R}(Q) \cap \mathscr{P}X$ and thus $x \in \bigcup (\mathfrak{R}(Q) \cap \mathscr{P}X)$. We have $X \subseteq \bigcup (\mathfrak{R}(Q) \cap \mathscr{P}X)$. □

3 Finite unions of Cartesian products

Let A, B be sets.

I will denote $\overline{X} = A \setminus X$.

Let denote $\Gamma(A; B)$ the set of all finite unions $X_0 \times Y_0 \cup \ldots \cup X_{n-1} \times Y_{n-1}$ of Cartesian products, where $n \in \mathbb{N}$ and $X_i \in \mathscr{P}A$, $Y_i \in \mathscr{P}B$ for every $i = 0, \ldots, n-1$.

Proposition 10. The following sets are pairwise equal:

1. $\Gamma(A; B)$;
2. the set of all sets of the form $\bigcup_{X \in S} (X \times Y_X)$ where S are finite collections on A and $Y_X \in \mathscr{P}B$ for every $X \in S$;
3. the set of all sets of the form $\bigcup_{X \in S} (X \times Y_X)$ where S are finite partitions of A and $Y_X \in \mathscr{P}B$ for every $X \in S$;
4. the set of all finite unions $\bigcup_{(X;Y) \in \sigma} (X \times Y)$ where σ is a relation between a partition of A and a partition of B (that is $\operatorname{dom} \sigma$ is a partition of A and $\operatorname{im} \sigma$ is a partition of B).
5. the set of all finite intersections $\bigcap_{i=0,\ldots,n-1} (X_i \times Y_i \cup \overline{X_i} \times B)$ where $n \in \mathbb{N}$ and $X_i \in \mathscr{P}A$, $Y_i \in \mathscr{P}B$ for every $i = 0, \ldots, n-1$.

Proof.

(1)⊇(2), (2)⊇(3). Obvious.

(1)⊆(2). Let $Q \in \Gamma(A; B)$. Then $Q = X_0 \times Y_0 \cup \ldots \cup X_{n-1} \times Y_{n-1}$. Denote $S = \{X_0, \ldots, X_{n-1}\}$. We have $Q = \bigcup_{X' \in S} (X' \times \bigcup \{Y_i \mid X_i = X'\}) \in (2)$.

(2)⊆(3). Let $Q = \bigcup_{X \in S} (X \times Y_X)$ where S is a finite collection on A and $Y_X \in \mathscr{P}B$ for every $X \in S$. Let

$$P = \bigcup_{X' \in \mathfrak{R}(S)} \left(X' \times \bigcup \{Y_X \mid X \in S \wedge X' \subseteq X\}\right)$$

To finish the proof let's show $P = Q$.

$\langle P \rangle^* \{x\} = \bigcup \{Y_X \mid X \in S \wedge X' \subseteq X\}$ where $x \in X'$.

Thus $\langle P \rangle^* \{x\} = \bigcup \{Y_X \mid X \in S \wedge x \in X\} = \langle Q \rangle^* \{x\}$. So $P = Q$.

(4)⊆(3). $\bigcup_{(X;Y) \in \sigma} (X \times Y) = \bigcup_{X \in \operatorname{dom} \sigma} (X \times \bigcup \{Y \in \mathscr{P}B \mid (X; Y) \in \sigma\}) \in (3)$.

(3)⊆(4). $\bigcup_{X \in S} (X \times Y_X) = \bigcup_{X \in S} (X \times \bigcup (\mathfrak{R}(\{Y_X \mid X \in S\}) \cap \mathscr{P}Y_X)) = \bigcup_{X \in S} (X \times \bigcup \{Y' \in \mathfrak{R}(\{Y_X \mid X \in S\}) \mid Y' \subseteq Y_X\}) = \bigcup_{X \in S} (X \times \bigcup \{Y' \in \mathfrak{R}(\{Y_X \mid X \in S\}) \mid (X; Y') \in \sigma\}) = \bigcup_{(X;Y) \in \sigma} (X \times Y)$ where σ is a relation between S and $\mathfrak{R}(\{Y_X \mid X \in S\})$, and $(X; Y') \in \sigma \Leftrightarrow Y' \subseteq Y_X$.

(5)⊆(3). Obvious.

(3)⊆(5). Let $Q = \bigcup_{X \in S} (X \times Y_X) = \bigcup_{i=0,\ldots,n-1} (X_i \times Y_i)$ for a partition $S = \{X_0, \ldots, X_{n-1}\}$ of A. Then $Q = \bigcap_{i=0,\ldots,n-1} (X_i \times Y_i \cup \overline{X_i} \times B)$. □

Exercise 1. Formulate the duals of these sets.

Proposition 11. $\Gamma(A; B)$ is a boolean lattice, a sublattice of the lattice $\mathscr{P}(A \times B)$.

Proof. That it's a sublattice is obvious. That it has complement, is also obvious. Distributivity follows from distributivity of $\mathscr{P}(A \times B)$. □

I will denote $\mathfrak{F}\Gamma(A;B) = \{(A;B;F) \mid F \in \mathfrak{F}\Gamma[A;B]\}$.

Remark 12. It should be instead be denoted as $(\mathfrak{F} \circ \Gamma)(A;B)$ but for brevity I omit $\circ$.

4 Before the diagram

Next we will prove the below theorem 35 (the theorem with a diagram). First we will present parts of this theorem as several lemmas, and then then state a statement about the diagram which concisely summarizes the lemmas (and their easy consequences).

Obvious 13. $\mathrm{up}^{\Gamma(\mathrm{Src}\, f;\mathrm{Dst}\, f)} f = (\mathrm{up}\, f) \cap \Gamma$ for every reloid f.

Conjecture 14. $\upuparrows^{\mathfrak{F}(\mathfrak{B})} \mathrm{up}^{\mathfrak{A}} \mathcal{X}$ is not a filter for some filter $\mathcal{X} \in \mathfrak{F}\Gamma(A;B)$ for some sets A, B.

Remark 15. About this conjecture see also:

- http://goo.gl/DHyuuU
- http://goo.gl/4a6wY6

Lemma 16. Let A, B be sets. The following are mutually inverse order isomorphisms between $\mathfrak{F}\Gamma(A;B)$ and $\mathsf{FCD}(A;B)$:

1. $\mathcal{A} \mapsto \bigsqcap^{\mathsf{FCD}} \mathrm{up}\, \mathcal{A}$;
2. $f \mapsto \mathrm{up}^{\Gamma(A;B)} f$.

Proof. Let's prove that $\mathrm{up}^{\Gamma(A;B)} f$ is a filter for every funcoid f. We need to prove that $P \cap Q \in \mathrm{up}\, f$ whenever

$$P = \bigcap_{i=0,\ldots,n-1} (X_i \times Y_i \cup \overline{X_i} \times B) \quad \text{and} \quad Q = \bigcap_{j=0,\ldots,m-1} (X'_j \times Y'_j \cup \overline{X'_j} \times B).$$

This follows from $P \in \mathrm{up}\, f \Leftrightarrow \forall i \in 0, \ldots, n-1\colon \langle f \rangle X_i \subseteq Y_i$ and likewise for Q, so having $\langle f \rangle (X_i \cap X'_j) \subseteq Y_i \cap Y'_j$ for every $i = 0, \ldots, n-1$ and $j = 0, \ldots, m-1$. From this it follows

$$((X_i \cap X'_j) \times (Y_i \cap Y'_j)) \cup (\overline{X_i \cap X'_j} \times B) \supseteq f$$

and thus $P \cap Q \in \mathrm{up}\, f$.

Let $\mathcal{A}$, $\mathcal{B}$ be filters on Γ. Let $\bigsqcap^{\mathsf{FCD}} \mathrm{up}\, \mathcal{A} = \bigsqcap^{\mathsf{FCD}} \mathrm{up}\, \mathcal{B}$. We need to prove $\mathcal{A} = \mathcal{B}$. (The rest follows from proof of the theorem 6.104 from my book [1]). We have: [TODO: Separate the first equality below from theorem 6.104 into a separate lemma.]

$$\mathcal{A} = \bigsqcap^{\mathsf{FCD}} \{X \times Y \cup \overline{X} \times B \in \mathcal{A} \mid X \in \mathscr{P}A, Y \in \mathscr{P}B\} =$$

$$\bigsqcap^{\mathsf{FCD}} \{X \times Y \cup \overline{X} \times B \mid X \in \mathscr{P}A, Y \in \mathscr{P}B, \exists P \in \mathcal{A} \colon P \subseteq X \times Y \cup \overline{X} \times B\} =$$

$$\bigsqcap^{\mathsf{FCD}} \{X \times Y \cup \overline{X} \times B \mid X \in \mathscr{P}A, Y \in \mathscr{P}B, \exists P \in \mathcal{A} \colon \langle P \rangle^{\ast} X \subseteq Y\} = (*)$$

$$\bigsqcap^{\mathsf{FCD}} \{X \times Y \cup \overline{X} \times B \mid X \in \mathscr{P}A, Y \in \mathscr{P}B, \bigsqcap \{\langle P \rangle^{\ast} X \mid A \in \mathrm{up}\, \mathcal{A}\} \sqsubseteq Y\} =$$

$$\bigsqcap^{\mathsf{FCD}} \left\{X \times Y \cup \overline{X} \times B \mid X \in \mathscr{P}A, Y \in \mathscr{P}B, \bigsqcap \left\{\langle P \rangle^{\ast} X \mid A \in \mathrm{up} \bigsqcap^{\mathsf{RLD}} \mathrm{up}\, \mathcal{A}\right\} \sqsubseteq Y\right\} =$$

$$\bigsqcap^{\mathsf{FCD}} \left\{X \times Y \cup \overline{X} \times B \mid X \in \mathscr{P}A, Y \in \mathscr{P}B, \left\langle (\mathsf{FCD}) \bigsqcap^{\mathsf{RLD}} \mathrm{up}\, \mathcal{A} \right\rangle X \sqsubseteq Y\right\} = (**)$$

$$\bigsqcap^{\mathsf{FCD}} \left\{X \times Y \cup \overline{X} \times B \mid X \in \mathscr{P}A, Y \in \mathscr{P}B, \left\langle \bigsqcap^{\mathsf{FCD}} \mathrm{up} \bigsqcap^{\mathsf{RLD}} \mathrm{up}\, \mathcal{A} \right\rangle X \sqsubseteq Y\right\} =$$

$$\bigsqcap^{\mathsf{FCD}} \left\{X \times Y \cup \overline{X} \times B \mid X \in \mathscr{P}A, Y \in \mathscr{P}B, \left\langle \bigsqcap^{\mathsf{FCD}} \mathrm{up}\, \mathcal{A} \right\rangle X \sqsubseteq Y\right\}.$$

(*) by properties of generalized filter bases, because $\{\langle P\rangle^* X \mid P \in \mathcal{A}\}$ is a filter base.

(**) by theorem 8.3 in [1].

Similarly

$$\operatorname{up}\mathcal{B} = \prod^{\mathsf{FCD}} \left\{ X \times Y \cup \overline{X} \times B \mid X \in \mathscr{P}A, Y \in \mathscr{P}B, \left\langle \prod^{\mathsf{FCD}} \operatorname{up}\mathcal{B} \right\rangle X \sqsubseteq Y \right\}.$$

Thus $\mathcal{A} = \mathcal{B}$. □

[TODO: For pointfree funcoids?]

Proposition 17. $g \circ f \in \Gamma(A; C)$ if $f \in \Gamma(A; B)$ and $g \in \Gamma(B; C)$ for some sets A, B, C.

Proof. Because composition of Cartesian products is a Cartesian product. □

Definition 18. $g \circ f = \prod^{\mathfrak{F}\Gamma(A;C)} \{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}$ for $f \in \mathfrak{F}\Gamma(A; B)$ and $g \in \mathfrak{F}\Gamma(B; C)$ (for every sets A, B, C).

We define f^{-1} for $f \in \mathfrak{F}\Gamma(A; B)$ similarly to f^{-1} for reloids and similarly derive the formulas:

1. $(f^{-1})^{-1} = f$;
2. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

4.1 Associativity over composition

I will denote base $(\mathfrak{A}; \mathfrak{Z}) = \mathfrak{A}$, core$(\mathfrak{A}; \mathfrak{Z}) = \mathfrak{Z}$ for a filtrator $(\mathfrak{A}; \mathfrak{Z})$. [TODO: move above in the book]

Obvious 19. $\mathscr{P}(\operatorname{core}\mathbb{F}) \cap \prod^{\mathfrak{F}(\operatorname{base}\mathbb{F})} \operatorname{up}^{\operatorname{base}\mathbb{F}} f = f$ for f ??.

Corollary 20. $\prod^{\mathfrak{F}(\operatorname{base}\mathbb{F})} \operatorname{up}^{\operatorname{base}\mathbb{F}}$ is an injection.

Lemma 21. $\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A;C)}(g \circ f) = \left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B;C)} g\right) \circ \left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B;C)}\right)$ for every $f \in \mathfrak{F}(\Gamma(A; B))$, $g \in \mathfrak{F}(\Gamma(B; C))$ (for every sets A, B, C).

Proof. If $K \in \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A;C)}(g \circ f)$ then $K \supseteq G \circ F$ for some $F \in f$, $G \in g$. But $F \in \operatorname{up}^{\Gamma(A;B)} f$, thus

$$F \in \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A;B)} f$$

and similarly

$$G \in \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B;C)} g.$$

So we have

$$K \supseteq G \circ F \in \left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B;C)} g \right) \circ \left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A;B)} f \right).$$

Let now

$$K \in \left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B;C)} g \right) \circ \left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A;B)} f \right).$$

Then there exist $F \in \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A;B)} f$ and $G \in \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B;C)} g$ such that $K \supseteq G \circ F$. By properties of generalized filter bases we can take $F \in \operatorname{up}^{\Gamma(A;B)} f$ and $G \in \operatorname{up}^{\Gamma(B;C)} g$. Thus $K \in \operatorname{up}^{\Gamma(A;C)}(g \circ f)$ and so $K \in \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A;C)}(g \circ f)$. □

Lemma 22. (FCD)$\prod^{\mathsf{RLD}} f = \prod^{\mathsf{FCD}} \operatorname{up} f$ for every $f \in \mathfrak{F}\Gamma(A; B)$ (where A, B are sets).

Proof. $(\mathsf{FCD})\bigsqcap^{\mathsf{RLD}} f = \bigsqcap^{\mathsf{FCD}} \mathrm{up} \bigsqcap^{\mathsf{RLD}} f = \bigsqcap^{\mathsf{FCD}} \mathrm{up}\, f$. □

Proposition 23. $(\mathsf{RLD})_{\mathrm{in}}(f \sqcup g) = (\mathsf{RLD})_{\mathrm{in}} f \sqcup (\mathsf{RLD})_{\mathrm{in}} g$ for every funcoids $f, g \in \mathsf{FCD}(A; B)$. [TODO: Move it above in the book.]

Proof. $(\mathsf{RLD})_{\mathrm{in}}(f \sqcup g) = \bigsqcup^{\mathsf{RLD}} \left\{ a \times^{\mathsf{RLD}} b \mid a \in \mathrm{atoms}^{\mathfrak{F}(A)}, b \in \mathrm{atoms}^{\mathfrak{F}(B)}, a \times^{\mathsf{FCD}} b \sqsubseteq f \sqcup g \right\} = \bigsqcup^{\mathsf{RLD}} \left\{ a \times^{\mathsf{RLD}} b \mid a \in \mathrm{atoms}^{\mathfrak{F}(A)}, b \in \mathrm{atoms}^{\mathfrak{F}(B)}, a \times^{\mathsf{FCD}} b \sqsubseteq f \vee a \times^{\mathsf{FCD}} b \sqsubseteq g \right\} = \bigsqcup^{\mathsf{RLD}} \left\{ a \times^{\mathsf{RLD}} b \mid a \in \mathrm{atoms}^{\mathfrak{F}(A)}, b \in \mathrm{atoms}^{\mathfrak{F}(B)}, a \times^{\mathsf{FCD}} b \sqsubseteq f \right\} \sqcup \bigsqcup^{\mathsf{RLD}} \left\{ a \times^{\mathsf{RLD}} b \mid a \in \mathrm{atoms}^{\mathfrak{F}(A)}, b \in \mathrm{atoms}^{\mathfrak{F}(B)}, a \times^{\mathsf{FCD}} b \sqsubseteq g \right\} = (\mathsf{RLD})_{\mathrm{in}} f \sqcup (\mathsf{RLD})_{\mathrm{in}} g$. □

Lemma 24. $(\mathsf{RLD})_{\mathrm{in}} X = X$ for $X \in \Gamma(A; B)$.

Proof. $X = X_0 \times Y_0 \cup \ldots \cup X_n \times Y_n = (X_0 \times^{\mathsf{FCD}} Y_0) \sqcup^{\mathsf{FCD}} \ldots \sqcup^{\mathsf{FCD}} (X_n \times^{\mathsf{FCD}} Y_n)$.

$(\mathsf{RLD})_{\mathrm{in}}\; X = (\mathsf{RLD})_{\mathrm{in}}(X_0 \times^{\mathsf{FCD}} Y_0) \sqcup^{\mathsf{RLD}} \ldots \sqcup^{\mathsf{RLD}} (\mathsf{RLD})_{\mathrm{in}}(X_n \times^{\mathsf{FCD}} Y) = (X_0 \times^{\mathsf{RLD}} Y_0) \sqcup^{\mathsf{RLD}} \ldots \sqcup^{\mathsf{RLD}} (X_n \times^{\mathsf{RLD}} Y_n) = X_0 \times Y_0 \cup \ldots \cup X_n \times Y_n = X$. □

Lemma 25. $\bigsqcap^{\mathsf{RLD}} \mathrm{up}\, f = (\mathsf{RLD})_{\mathrm{in}} \bigsqcap^{\mathsf{FCD}} \mathrm{up}\, f$ for every filter $f \in \mathfrak{F}\Gamma(A; B)$.

Proof. $(\mathsf{RLD})_{\mathrm{in}} \bigsqcap^{\mathsf{FCD}} f = \bigsqcap^{\mathsf{RLD}} \langle (\mathsf{RLD})_{\mathrm{in}} \rangle^* \mathrm{up}\, f = (\text{by the previous lemma}) = \bigsqcap^{\mathsf{RLD}} \mathrm{up}\, f$. □

Lemma 26.

1. $f \mapsto \bigsqcap^{\mathsf{RLD}} \mathrm{up}\, f$ and $\mathcal{A} \mapsto \Gamma(A; B) \cap \mathrm{up}\, \mathcal{A}$ are mutually inverse bijections between $\mathfrak{F}\Gamma(A; B)$ and a subset of reloids.
2. These bijections preserve composition.

Proof. 1. That they are mutually inverse bijections is obvious.

2. $\left(\bigsqcap^{\mathsf{RLD}} \mathrm{up}\, g \right) \circ \left(\bigsqcap^{\mathsf{RLD}} \mathrm{up}\, f \right) = \bigsqcap^{\mathsf{RLD}} \left\{ G \circ F \mid F \in \bigsqcap^{\mathsf{RLD}} f, G \in \bigsqcap^{\mathsf{RLD}} g \right\} = \bigsqcap^{\mathsf{RLD}} \{ G \circ F \mid F \in f, G \in g \} = \bigsqcap^{\mathsf{RLD}} \bigsqcap^{\mathfrak{F}\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, g)} \{ G \circ F \mid F \in f, G \in g \} = \bigsqcap^{\mathsf{RLD}} (g \circ f)$. So $\bigsqcap^{\mathsf{RLD}}$ preserves composition. That $\mathcal{A} \mapsto \Gamma(A; B) \cap \mathrm{up}\, \mathcal{A}$ preserves composition follows from properties of bijections. □

Lemma 27. Let A, B, C be sets.

1. $\left(\bigsqcap^{\mathsf{FCD}} \mathrm{up}\, g \right) \circ \left(\bigsqcap^{\mathsf{FCD}} \mathrm{up}\, f \right) = \bigsqcap^{\mathsf{FCD}} \mathrm{up}(g \circ f)$ for every $f \in \mathfrak{F}\Gamma(A; B)$, $g \in \mathfrak{F}\Gamma(B; C)$;
2. $\left(\mathrm{up}^{\Gamma(B;C)}\, g \right) \circ \left(\mathrm{up}^{\Gamma(A;B)}\, f \right) = \mathrm{up}^{\Gamma(A;B)}(g \circ f)$ for every funcoids $f \in \mathsf{FCD}(A;\, B)$ and $g \in \mathsf{FCD}(B\colon C)$.

Proof. It's enough to prove only the first formula, because of the bijection from thereom 16.

Really: $\bigsqcap^{\mathsf{FCD}} \mathrm{up}(g \circ f) = \bigsqcap^{\mathsf{FCD}} \mathrm{up} \bigsqcap^{\mathsf{RLD}} \mathrm{up}(g \circ f) = \bigsqcap^{\mathsf{FCD}} \mathrm{up}\left(\bigsqcap^{\mathsf{RLD}} \mathrm{up}\, g \circ \bigsqcap^{\mathsf{RLD}} \mathrm{up}\, f \right) = (\mathsf{FCD})\left(\bigsqcap^{\mathsf{RLD}} \mathrm{up}\, g \circ \bigsqcap^{\mathsf{RLD}} \mathrm{up}\, f \right) = \left((\mathsf{FCD}) \bigsqcap^{\mathsf{RLD}} \mathrm{up}\, g \right) \circ \left((\mathsf{FCD}) \bigsqcap^{\mathsf{RLD}} \mathrm{up}\, f \right) = \left(\bigsqcap^{\mathsf{FCD}} \mathrm{up} \bigsqcap^{\mathsf{RLD}} \mathrm{up}\, g \right) \circ \left(\bigsqcap^{\mathsf{FCD}} \mathrm{up} \bigsqcap^{\mathsf{RLD}} \mathrm{up}\, f \right) = \left(\bigsqcap^{\mathsf{FCD}} \mathrm{up}\, g \right) \circ \left(\bigsqcap^{\mathsf{FCD}} \mathrm{up}\, f \right)$. □

Corollary 28. $(h \circ g) \circ f = h \circ (g \circ f)$ for every $f \in \mathfrak{F}(\Gamma(A; B))$, $g \in \mathfrak{F}\Gamma(B; C)$, $h \in \mathfrak{F}\Gamma(C; D)$ for every sets A, B, C, D.

Lemma 29. $\Gamma(A; B) \cap \mathrm{GR}\, f$ is a filter on the lattice $\Gamma(A; B)$ for every reloid $f \in \mathsf{RLD}(A; B)$.

Proof. That it is an upper set, is obvious. If $A, B \in \Gamma(A; B) \cap \mathrm{GR}\, f$ then $A, B \in \Gamma(A; B)$ and A, $B \in \mathrm{GR}\, f$. Thus $A \cap B \in \Gamma(A; B) \cap \mathrm{GR}\, f$. □

Proposition 30. If $Y \in \mathrm{up}\, \langle f \rangle \mathcal{X}$ for a funcoid f then there exists $A \in \mathrm{up}\, \mathcal{X}$ such that $Y \in \mathrm{up}\, \langle f \rangle A$.

Proof. $Y \in \mathrm{up} \bigsqcap^{\mathfrak{F}} \{ \langle f \rangle A \mid A \in \mathrm{up}\, a \}$.

So by properties of generalized filter bases, there exists $A \in \operatorname{up} a$ such that $Y \in \operatorname{up} \langle f \rangle A$. □

Lemma 31. $(\mathsf{FCD}) f = \bigsqcap^{\mathsf{FCD}} (\Gamma(A; B) \cap \operatorname{GR} f)$ for every reloid $f \in \mathsf{RLD}(A; B)$.

Proof. Let a be an ultrafilter. We need to prove

$$\langle (\mathsf{FCD}) f \rangle a = \left\langle \bigsqcap^{\mathsf{FCD}} (\Gamma(A; B) \cap \operatorname{GR} f) \right\rangle a$$

that is

$$\left\langle \bigsqcap^{\mathsf{FCD}} \operatorname{up} f \right\rangle a = \left\langle \bigsqcap^{\mathsf{FCD}} (\Gamma(A; B) \cap \operatorname{GR} f) \right\rangle a$$

that is

$$\bigsqcap^{\mathfrak{F}}_{F \in \operatorname{up} f} \langle F \rangle a = \bigsqcap^{\mathfrak{F}}_{F \in \Gamma(A; B) \cap \operatorname{up} f} \langle F \rangle a.$$

For this it's enough to prove that $Y \in \operatorname{up} \langle F \rangle a$ for some $F \in \operatorname{up} f$ implies $Y \in \operatorname{up} \langle F' \rangle a$ for some $F' \in \Gamma(A; B) \cap \operatorname{GR} f$.

Let $Y \in \operatorname{up} \langle F \rangle a$. Then (proposition above) there exists $A \in \operatorname{up} a$ such that $Y \in \operatorname{up} \langle F \rangle A$.

$Y \in \operatorname{up} \langle A \times^{\mathsf{FCD}} Y \sqcup \overline{A} \times^{\mathsf{FCD}} 1 \rangle a$; $\langle A \times^{\mathsf{FCD}} Y \sqcup \overline{A} \times^{\mathsf{FCD}} 1 \rangle \mathcal{X} = Y \in \operatorname{up} \langle F \rangle \mathcal{X}$ if $0 \neq \mathcal{X} \sqsubseteq A$ and $\langle A \times^{\mathsf{FCD}} Y \sqcup \overline{A} \times^{\mathsf{FCD}} 1 \rangle \mathcal{X} = 1 \in \operatorname{up} \langle F \rangle \mathcal{X}$ if $\mathcal{X} \not\sqsubseteq A$.

Thus $A \times^{\mathsf{FCD}} Y \sqcup \overline{A} \times^{\mathsf{FCD}} 1 \sqsupseteq F$. So $A \times^{\mathsf{FCD}} Y \sqcup \overline{A} \times^{\mathsf{FCD}} 1$ is the sought for F'. □

4.2 Relationships between (FCD) and $(\mathsf{RLD})_\Gamma$

Definition 32. $(\mathsf{RLD})_\Gamma f = \bigsqcap^{\mathsf{RLD}} \operatorname{up}^{\Gamma(\operatorname{Src} f; \operatorname{Dst} f)} f$ for every funcoid f. I call $(\mathsf{RLD})_\Gamma$ as Γ-reloid or Gamma-reloid.

Lemma 33. $(\mathsf{FCD})(\mathsf{RLD})_\Gamma f = f$ for every funcoid f.

Proof. For every filter $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$ we have $\langle (\mathsf{FCD})(\mathsf{RLD})_\Gamma f \rangle \mathcal{X} = \bigsqcap^{\mathfrak{F}}_{F \in \operatorname{up} (\mathsf{RLD})_\Gamma f} \langle F \rangle \mathcal{X} = \bigsqcap^{\mathfrak{F}}_{F \in \operatorname{up}^{\Gamma(\operatorname{Src} f; \operatorname{Dst} f)} f} \langle F \rangle \mathcal{X}$.

Obviously $\bigsqcap^{\mathfrak{F}}_{F \in \operatorname{up}^{\Gamma(\operatorname{Src} f; \operatorname{Dst} f)} f} \langle F \rangle \mathcal{X} \sqsupseteq \langle f \rangle \mathcal{X}$. So $(\mathsf{FCD})(\mathsf{RLD})_\Gamma f \sqsupseteq f$.

Let $Y \in \operatorname{up} \langle f \rangle \mathcal{X}$. Then (propositiona above) there exists $A \in \operatorname{up} \mathcal{X}$ such that $Y \in \operatorname{up} \langle f \rangle A$.

Thus $A \times Y \cup \overline{A} \times 1 \in \operatorname{up} f$. So $\langle (\mathsf{FCD})(\mathsf{RLD})_\Gamma f \rangle \mathcal{X} = \bigsqcap^{\mathfrak{F}}_{F \in \operatorname{up}^{\Gamma(\operatorname{Src} f; \operatorname{Dst} f)} f} \langle F \rangle \mathcal{X} \sqsubseteq \langle A \times Y \cup \overline{A} \times 1 \rangle \mathcal{X} = Y$. So $Y \in \operatorname{up} \langle (\mathsf{FCD})(\mathsf{RLD})_\Gamma f \rangle \mathcal{X}$ that is $\langle f \rangle \mathcal{X} \sqsupseteq \langle (\mathsf{FCD})(\mathsf{RLD})_\Gamma f \rangle \mathcal{X}$ that is $f \sqsupseteq (\mathsf{FCD})(\mathsf{RLD})_\Gamma f$. □

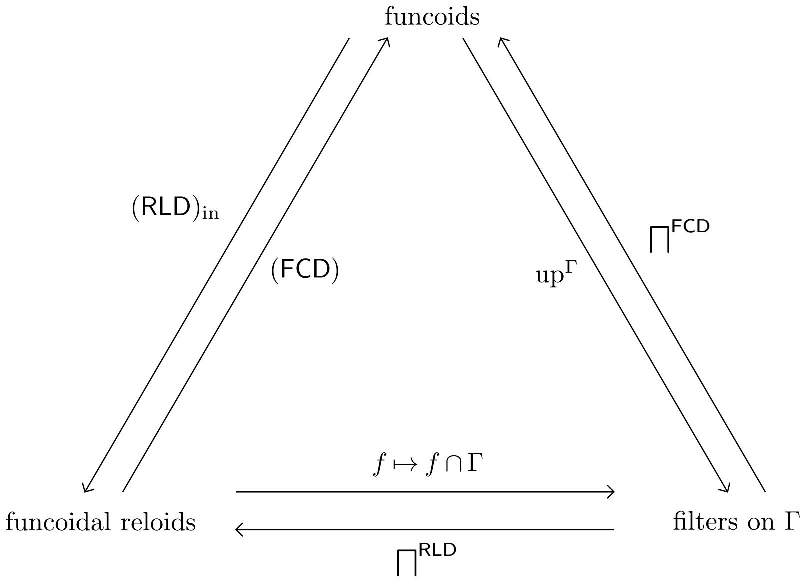
Proposition 34. $(\mathsf{RLD})_\Gamma$ is neither upper nor lower adjoint of (FCD) (in general).

Proof. It is not upper adjoint because $(\mathsf{RLD})_{\mathrm{in}}$ is the upper adjoint of (FCD) and $(\mathsf{RLD})_{\mathrm{in}} \neq (\mathsf{RLD})_\Gamma$.

If $(\mathsf{RLD})_\Gamma$ is the lower adjoint of (FCD), then $f \sqsupseteq (\mathsf{RLD})_\Gamma (\mathsf{FCD}) f$ and thus $f \sqsupseteq (\mathsf{RLD})_{\mathrm{in}} (\mathsf{FCD}) f$. But $f \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} (\mathsf{FCD}) f$, thus having $(\mathsf{RLD})_{\mathrm{in}} (\mathsf{FCD}) f = f$ what is not an identity (take $f = (=)|_A$ for an infinite set A). □

5 The diagram

Theorem 35. The following is a commutative diagram (in category **Set**), every arrow in this diagram is an isomorphism. Every cycle in this diagram is an identity (therefore "parallel" arrows are mutually inverse). The arrows preserve order, composition, and reversal ($f \mapsto f^{-1}$).



Proof. First we need to show that $\bigsqcap^{\mathsf{RLD}} f$ is a funcoidal reloid. But it follows from lemma 25.

Next, we need to show that all morphisms depicted on the diagram are bijections and the depicted “opposite” morphisms are mutually inverse.

That (FCD) and $(\mathsf{RLD})_{\mathrm{in}}$ are mutually inverse was proved above in the book.

That $\bigsqcap^{\mathsf{RLD}}$ and $f \mapsto f \cap \Gamma$ are mutually inverse was proved above.

That $\bigsqcap^{\mathsf{FCD}}$ and up^{Γ} are mutually inverse was proved above.

It remains to prove that three-element cycles are identities. But this follows from lemma 25.

That the morphisms preserve order and composition was proved above. That they preserve reversal is obvious. □

6 Some additional properties

Proposition 36. For every funcoid $f \in \mathsf{FCD}(A; B)$ (for sets A, B):

1. $\operatorname{dom} f = \bigsqcap^{\mathfrak{F}(A)} \langle \operatorname{dom} \rangle^{*} \mathrm{up}^{\Gamma(A;B)} f$;
2. $\operatorname{im} f = \bigsqcap^{\mathfrak{F}(A)} \langle \operatorname{im} \rangle^{*} \mathrm{up}^{\Gamma(A;B)} f$.

Proof. Take $\{X \times Y \mid X \in \mathscr{P}A, Y \in \mathscr{P}A, X \times Y \sqsupseteq f\} \subseteq \mathrm{up}^{\Gamma(A;B)} f$. I leave the rest reasoning as an exercise. □

Proposition 37. $(\mathsf{RLD})_{\Gamma} f \sqsupseteq (\mathsf{RLD})_{\mathrm{in}} f \sqsupseteq (\mathsf{RLD})_{\mathrm{out}} f$ for every funcoid f.

Proof. We already know that $(\mathsf{RLD})_{\mathrm{in}} f \sqsupseteq (\mathsf{RLD})_{\mathrm{out}} f$ (see above in the book).

The formula $(\mathsf{RLD})_{\Gamma} f \sqsupseteq (\mathsf{RLD})_{\mathrm{in}} f$ follows from $\forall G \in \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)} f: G \sqsupseteq f$. □

Example 38. $(\mathsf{RLD})_{\Gamma} f \sqsupset (\mathsf{RLD})_{\mathrm{in}} f \sqsupset (\mathsf{RLD})_{\mathrm{out}} f$ for some funcoid f.

Proof. Take $f = (=)|_{\mathbb{R}}$. We already know that $(\mathsf{RLD})_{\mathrm{in}} f \sqsupset (\mathsf{RLD})_{\mathrm{out}} f$ (see above in the book).

It remains to prove $(\mathsf{RLD})_{\Gamma} f \neq (\mathsf{RLD})_{\mathrm{in}} f$.

Take $F = \bigcup_{i \in \mathbb{Z}} ([i; i+1[\times [i; i+1[)$.

Then $F \in f = \mathrm{up}\, (\mathsf{RLD})_{\mathrm{in}} f$ (because $\langle F \rangle a \sqsupseteq \langle f \rangle a$ for both principal ultrafilter $a = \{i\}$ and every other ultrafilter a).

It remains to prove $F \notin \mathrm{up}\, (\mathsf{RLD})_{\Gamma} f$.

Suppose $F \in \mathrm{up}\,(\mathsf{RLD})_\Gamma = \mathrm{up} \bigsqcap^{\mathsf{RLD}} \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f$. Then by properties of generalized filter bases, there is $F' \in \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f$ such that $F \sqsupseteq F'$. Because $F' \subseteq \bigcup_{i \in \mathbb{Z}} ([i; i+1[\times [i; i+1[)$ and $F' \sqsupseteq (=)|_{\mathbb{R}}$, there is a point $q \in [i; i+1[\times [i; i+1[$ for each $i \in \mathbb{Z}$; thus, $F' \notin \Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)$.

Thus $F \notin \mathrm{up}\,(\mathsf{RLD})_\Gamma\, f$. □

Theorem 39. For every reloid f and $\mathcal{X} \in \mathfrak{F}(\mathrm{Src}\, f)$, $\mathcal{Y} \in \mathfrak{F}(\mathrm{Dst}\, f)$:

1. $\mathcal{X}\, [(\mathsf{FCD})\, f]\, \mathcal{Y} \Leftrightarrow \forall F \in \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f \colon \mathcal{X}\, [F]\, \mathcal{Y}$;
2. $\langle (\mathsf{FCD})\, f \rangle \mathcal{X} = \bigsqcap^{\mathfrak{F}}_{F \in \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f} \langle F \rangle \mathcal{X}$.

Proof. 1. $\forall F \in \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f \colon \mathcal{X}\, [F]\, \mathcal{Y} \Leftrightarrow$ (by properties of generalized filter bases, taking into account that funcoids are isomorphic to filters)$\Leftrightarrow \mathcal{X} \left[\bigsqcap^{\mathsf{FCD}} \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f\right] \mathcal{Y} \Leftrightarrow \mathcal{X}\, [(\mathsf{FCD})\, f]\, \mathcal{Y}$.

2. $\bigsqcap^{\mathfrak{F}}_{F \in \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f} \langle F \rangle a = \left\langle \bigsqcap^{\mathsf{FCD}} \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f \right\rangle a = \langle (\mathsf{FCD})\, f \rangle a$ for every ultrafilter a.

It remains to prove that the function

$$\varphi = \lambda \mathcal{X} \in \mathfrak{F}(\mathrm{Src}\, f) \colon \bigsqcap^{\mathfrak{F}}_{F \in \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f} \langle F \rangle \mathcal{X}$$

is a component of a funcoid (from what follows that $\varphi = \langle (\mathsf{FCD})\, f \rangle$). To prove this, it's enough to show that it preserves finite joins and filtered meets. [TODO: Definition of filtered meets.]

$\varphi 0 = 0$ is obvious. $\varphi(\mathcal{I} \sqcup \mathcal{J}) = \bigsqcap^{\mathfrak{F}}_{F \in \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f} (\langle F \rangle \mathcal{I} \sqcup \langle F \rangle \mathcal{J}) = \bigsqcap^{\mathfrak{F}}_{F \in \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f} \langle F \rangle \mathcal{I} \sqcup \bigsqcap^{\mathfrak{F}}_{F \in \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f} \langle F \rangle \mathcal{J} = \varphi \mathcal{I} \sqcup \varphi \mathcal{J}$. If S is a generalized filter base of $\mathrm{Src}\, f$, then $\varphi \bigsqcap^{\mathfrak{F}} S = \bigsqcap^{\mathfrak{F}}_{F \in \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f} \langle F \rangle \bigsqcap^{\mathfrak{F}} S = \bigsqcap^{\mathfrak{F}}_{F \in \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f} \bigsqcap^{\mathfrak{F}} \langle \langle F \rangle \rangle^* S = \bigsqcap^{\mathfrak{F}}_{F \in \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f} \bigsqcap^{\mathfrak{F}}_{\mathcal{X} \in S} \langle F \rangle \mathcal{X} = \bigsqcap^{\mathfrak{F}}_{\mathcal{X} \in S} \bigsqcap^{\mathfrak{F}}_{F \in \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)}\, f} \langle F \rangle \mathcal{X} = \bigsqcap^{\mathfrak{F}}_{\mathcal{X} \in S} \varphi \mathcal{X} = \bigsqcap^{\mathfrak{F}} \langle \varphi \rangle^* S$.

So φ is a component of a funcoid. □

Definition 40. $\boxdot f = \bigsqcap^{\mathsf{RLD}} \mathrm{up}^{\Gamma(\mathrm{Src}\, f; \mathrm{Dst}\, f)} f$ for reloid f.

Conjecture 41. For every reloid f:

1. $\boxdot f = (\mathsf{RLD})_{\mathrm{in}}\, (\mathsf{FCD})\, f$;
2. $\boxdot f = (\mathsf{RLD})_\Gamma\, (\mathsf{FCD})\, f$.

Obvious 42. $\boxdot f \sqsupseteq f$ for every reloid f.

Example 43. $(\mathsf{RLD})_\Gamma f \neq \boxdot (\mathsf{RLD})_{\mathrm{out}} f$ for some funcoid f.

Proof. Take $f = \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{N})}$. Then, as it was shown above, $(\mathsf{RLD})_{\mathrm{out}} f = 0$ and thus $\boxdot (\mathsf{RLD})_{\mathrm{out}} f = 0$. But $(\mathsf{RLD})_\Gamma f \sqsupseteq (\mathsf{RLD})_{\mathrm{in}}\, f \neq 0$. So $(\mathsf{RLD})_\Gamma f \neq \boxdot (\mathsf{RLD})_{\mathrm{out}} f$. □

Conjecture 44. $(\mathsf{RLD})_\Gamma f = \boxdot (\mathsf{RLD})_{\mathrm{in}} f$ for every funcoid f.

Proposition 45. [TODO: Move it above in the book.] $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Leftrightarrow \mathrm{dom}\, f \sqsubseteq \mathcal{A} \wedge \mathrm{im}\, f \sqsubseteq \mathcal{B}$ for every funcoid f and filters $\mathcal{A} \in \mathfrak{F}(\mathrm{Src}\, f)$, $\mathcal{B} \in \mathfrak{F}(\mathrm{Dst}\, f)$.

Proof. $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Rightarrow \mathrm{dom}\, f \sqsubseteq \mathcal{A}$ because $\mathrm{dom}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \sqsubseteq \mathcal{A}$.

Let now $\mathrm{dom}\, f \sqsubseteq \mathcal{A} \wedge \mathrm{im}\, f \sqsubseteq \mathcal{B}$. Then $\langle f \rangle \mathcal{X} \neq 0 \Rightarrow \mathcal{X} \not\asymp \mathcal{A}$ that is $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} 1$. Similarly $f \sqsubseteq 1 \times^{\mathsf{FCD}} \mathcal{B}$. Thus $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. □

Theorem 46. $\mathrm{dom}\, (\mathsf{RLD})_{\mathrm{in}} f = \mathrm{dom}\, f$ and $\mathrm{im}\, (\mathsf{RLD})_{\mathrm{in}} f = \mathrm{im}\, f$ for every funcoid f. [TODO: Move it above in the book, remove the conjecture which this statement proves.]

Proof. We have for every filter $\mathcal{X} \in \mathfrak{F}(\mathrm{Src}\, f)$:

$\mathcal{X} \sqsupseteq \mathrm{dom}\, (\mathsf{RLD})_{\mathrm{in}} f \Leftrightarrow \mathcal{X} \times^{\mathsf{RLD}} 1 \sqsupseteq (\mathsf{RLD})_{\mathrm{in}} f \Leftrightarrow \forall a \in \mathfrak{F}(\mathrm{Src}\, f), b \in \mathfrak{F}(\mathrm{Dst}\, f) \colon (a \times^{\mathsf{FCD}} b \sqsubseteq f \Rightarrow a \times^{\mathsf{RLD}} b \sqsubseteq \mathcal{X} \times^{\mathsf{RLD}} 1) \Leftrightarrow \forall a \in \mathfrak{F}(\mathrm{Src}\, f), b \in \mathfrak{F}(\mathrm{Dst}\, f) \colon (a \times^{\mathsf{FCD}} b \sqsubseteq f \Rightarrow a \sqsubseteq \mathcal{X})$;

$\mathcal{X} \sqsupseteq \operatorname{dom} f \Leftrightarrow \mathcal{X} \times^{\mathsf{RLD}} 1 \sqsupseteq f \Leftrightarrow \mathcal{X} \times^{\mathsf{FCD}} 1 \sqsupseteq f \Leftrightarrow \forall a \in \mathfrak{F}(\operatorname{Src} f), b \in \mathfrak{F}(\operatorname{Dst} f) \colon (a \times^{\mathsf{FCD}} b \sqsubseteq f \Rightarrow a \times^{\mathsf{FCD}} b \sqsubseteq \mathcal{X} \times^{\mathsf{FCD}} 1) \Leftrightarrow \forall a \in \mathfrak{F}(\operatorname{Src} f), b \in \mathfrak{F}(\operatorname{Dst} f) \colon (a \times^{\mathsf{FCD}} b \sqsubseteq f \Rightarrow a \sqsubseteq \mathcal{X})$.

Thus $\operatorname{dom} (\mathsf{RLD})_{\mathrm{in}} f = \operatorname{dom} f$. The rest follows from symmetry. □

Proposition 47. $\operatorname{dom} (\mathsf{RLD})_\Gamma f = \operatorname{dom} f$ and $\operatorname{im} (\mathsf{RLD})_\Gamma f = \operatorname{im} f$ for every funcoid f.

Proof. $\operatorname{dom} (\mathsf{RLD})_\Gamma f \sqsupseteq \operatorname{dom} f$ and $\operatorname{im} (\mathsf{RLD})_\Gamma f \sqsupseteq \operatorname{im} f$ because $(\mathsf{RLD})_\Gamma f \sqsupseteq (\mathsf{RLD})_{\mathrm{in}}$ and $\operatorname{dom} (\mathsf{RLD})_{\mathrm{in}} f = \operatorname{dom} f$ and $\operatorname{im} (\mathsf{RLD})_{\mathrm{in}} f = \operatorname{im} f$.

It remains to prove (as the rest follows from symmetry) that $\operatorname{dom} (\mathsf{RLD})_\Gamma f \sqsubseteq \operatorname{dom} f$.

Really, $\operatorname{dom} (\mathsf{RLD})_\Gamma f \sqsubseteq \bigsqcap^{\mathfrak{F}} \{X \in \operatorname{up} \operatorname{dom} f \mid X \times 1 \in \operatorname{up} f\} = \bigsqcap^{\mathfrak{F}} \{X \in \operatorname{up} \operatorname{dom} f \mid X \in \operatorname{up} \operatorname{dom} f\} = \bigsqcap^{\mathfrak{F}} \operatorname{up} \operatorname{dom} f = \operatorname{dom} f$. □

Conjecture 48. For every funcoid g we have $\operatorname{Cor} (\mathsf{RLD})_\Gamma g = (\mathsf{RLD})_\Gamma \operatorname{Cor} g$.

7 More on properties of funcoids

Proposition 49. $\Gamma(A; B)$ is the center of lattice $\mathsf{FCD}(A; B)$.

Proof. See theorem 4.139 in [1]. □

Proposition 50. $\operatorname{up}^{\Gamma(A;B)} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$ is defined by the filter base $\{A \times B \mid A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}\}$ on the lattice $\Gamma(A; B)$.

Proof. It follows from the fact that $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} = \bigsqcap^{\mathsf{FCD}} \{A \times B \mid A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}\}$. □

Proposition 51. $\operatorname{up}^{\Gamma(A;B)} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathfrak{F}(\Gamma(A; B)) \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$.

Proof. It follows from the fact that $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} = \bigsqcap^{\mathsf{FCD}} \{A \times B \mid A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}\}$. □

Proposition 52. For every $f \in \mathfrak{F}(\Gamma(A; B))$:

1. $f \circ f$ is defined by the filter base $\{F \circ F \mid F \in \operatorname{up} f\}$ (if $A = B$);
2. $f^{-1} \circ f$ is defined by the filter base $\{F^{-1} \circ F \mid F \in \operatorname{up} f\}$;
3. $f \circ f^{-1}$ is defined by the filter base $\{F \circ F^{-1} \mid F \in \operatorname{up} f\}$.

Proof. I will prove only (1) and (2) because (3) is analogous to (2).

1. It's enough to show that $\forall F, G \in \operatorname{up} f \exists H \in \operatorname{up} f \colon H \circ H \sqsubseteq G \circ F$. To prove it take $H = F \sqcap G$.
2. It's enough to show that $\forall F, G \in \operatorname{up} f \exists H \in \operatorname{up} f \colon H^{-1} \circ H \sqsubseteq G^{-1} \circ F$. To prove it take $H = F \sqcap G$. Then $H^{-1} \circ H = (F \sqcap G)^{-1} \circ (F \sqcap G) \sqsubseteq G^{-1} \circ F$. □

Theorem 53. For every sets A, B, C if $g, h \in \mathfrak{F}\Gamma(A; B)$ then

1. $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$;
2. $(g \sqcup h) \circ f = g \circ f \sqcup h \circ f$.

Proof. It follows from the order isomorphism above, which preserves composition. □

Bibliography

[1] Victor Porton. *Algebraic General Topology. Volume 1*. 2014.