Pointfree Funcoids*

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Abstract

It is a part of my Algebraic General Topology research.

I generalize the point-set notion of funcoids to pointfree topology notion of *pointfree funcoids*.

In my yet unpublished research pointfree funcoids are used to define products of funcoids.

Keywords: algebraic general topology, funcoids, reloids

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1 Preface

This article generalizes the notions of funcoids introduced in [2], I call this generalization *pointfree funcoids*.

In my yet unpublished research pointfree funcoids are used to define products of funcoids.

2 Notation

First we use the notation introduced in [3] and [2].

*. This document has been written using the GNU $\text{T}_\text{E}\text{X}_{\text{MACS}}$ text editor (see `www.texmacs.org`).

In addition to $\operatorname{atoms}^{\mathfrak{A}} x$, the set of atoms under an element x of a poset $\mathfrak{A}$ we will write just $\operatorname{atoms}^{\mathfrak{A}}$ for all atoms of a poset $\mathfrak{A}$.

3 Pointfree funcoids

3.1 Definition

Definition 1. *Pointfree funcoid* is a quadruple $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$ where $\mathfrak{A}$ and $\mathfrak{B}$ are posets, $\alpha \in \mathfrak{B}^{\mathfrak{A}}$ and $\beta \in \mathfrak{A}^{\mathfrak{B}}$ such that

$$\forall x \in \mathfrak{A}, y \in \mathfrak{B}: (y \not\asymp^{\mathfrak{B}} \alpha x \Leftrightarrow x \not\asymp^{\mathfrak{A}} \beta y).$$

Definition 2. The *source* $\operatorname{Src}(\mathfrak{A}; \mathfrak{B}; \alpha; \beta) = \mathfrak{A}$ and *destination* $\operatorname{Dst}(\mathfrak{A}; \mathfrak{B}; \alpha; \beta) = \mathfrak{B}$ for every pointfree funcoid $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$.

Definition 3. I will denote $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ the set of pointfree funcoids from $\mathfrak{A}$ to $\mathfrak{B}$ (that is with source $\mathfrak{A}$ and destination $\mathfrak{B}$), for every posets $\mathfrak{A}$ and $\mathfrak{B}$.

Proposition 4. If $\mathfrak{A}$ and $\mathfrak{B}$ have least elements, then $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ has least element.

Proof. It is $(\mathfrak{A}; \mathfrak{B}; \mathfrak{A} \times \{0^{\mathfrak{B}}\}; \mathfrak{B} \times \{0^{\mathfrak{A}}\})$. □

Definition 5. $\langle(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)\rangle \overset{\text{def}}{=} \alpha$ for a pointfree funcoid $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$.

Definition 6. $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)^{-1} = (\mathfrak{B}; \mathfrak{A}; \beta; \alpha)$ for a pointfree funcoid $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$.

Proposition 7. If f is a pointfree funcoid then f^{-1} is also a pointfree funcoid.

Proof. Follows from symmetry in the definition of pointfree funcoid. □

Obvious 8. $(f^{-1})^{-1} = f$ for a pointfree funcoid f.

Definition 9. The relation $[f] \in \mathscr{P}(\operatorname{Src} f \times \operatorname{Dst} f)$ is defined by the formula (for every pointfree funcoid f and $x \in \operatorname{Src} f$, $y \in \operatorname{Dst} f$)

$$x\, [f]\, y \overset{\text{def}}{=} y \not\asymp^{\operatorname{Dst} f} \langle f \rangle x.$$

Obvious 10. $x\, [f]\, y \Leftrightarrow y \not\asymp^{\operatorname{Dst} f} \langle f \rangle x \Leftrightarrow x \not\asymp^{\operatorname{Src} f} \langle f^{-1} \rangle y$ for every pointfree funcoid f and $x \in \operatorname{Src} f$, $y \in \operatorname{Dst} f$.

Obvious 11. $[f^{-1}] = [f]^{-1}$ for a pointfree funcoid f.

Theorem 12. Let $\mathfrak{A}$ and $\mathfrak{B}$ are posets. Then:

1. If $\mathfrak{A}$ is separable, for given value of $\langle f \rangle$ exists no more than one $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$;
2. If $\mathfrak{A}$ and $\mathfrak{B}$ are separable, for given value of $[f]$ exists no more than one $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.

Proof. Let $f, g \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.

1. Let $\langle f \rangle = \langle g \rangle$. Then for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$ we have $x \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle y \Leftrightarrow y \not\asymp^{\mathfrak{B}} \langle f \rangle x \Leftrightarrow y \not\asymp^{\mathfrak{B}} \langle g \rangle x \Leftrightarrow x \not\asymp^{\mathfrak{A}} \langle g^{-1} \rangle y$ and thus by separability of $\mathfrak{A}$ we have $\langle f^{-1} \rangle y = \langle g^{-1} \rangle y$ that is $\langle f^{-1} \rangle = \langle g^{-1} \rangle$ and so $f = g$.
2. Let $[f] = [g]$. Then for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$ we have $y \not\asymp^{\mathfrak{B}} \langle f \rangle x \Leftrightarrow x\, [f]\, y \Leftrightarrow x\, [g]\, y \Leftrightarrow y \not\asymp^{\mathfrak{B}} \langle g \rangle x$ and thus by separability of $\mathfrak{B}$ we have $\langle f \rangle x = \langle g \rangle x$ that is $\langle f \rangle = \langle g \rangle$. Similarly we have $\langle f^{-1} \rangle = \langle g^{-1} \rangle$. Thus $f = g$. □

Proposition 13. If $\operatorname{Dst} f$ is separable, then $\langle f \rangle 0^{\operatorname{Src} f} = 0^{\operatorname{Dst} f}$ for every pointfree funcoid f.

Proof. $y \not\asymp \langle f \rangle 0^{\operatorname{Src} f} \Leftrightarrow 0^{\operatorname{Src} f} \not\asymp \langle f^{-1} \rangle y \Leftrightarrow 0 \Leftrightarrow y \not\asymp 0^{\operatorname{Dst} f}$. Thus by separability, $\langle f \rangle 0^{\operatorname{Src} f} = 0^{\operatorname{Dst} f}$. □

Proposition 14. If $\operatorname{Dst} f$ is a separable meet semilattice with least element then $\langle f \rangle$ is a monotone function (for a pointfree funcoid f).

Proof. $a \subseteq b \Rightarrow \forall x \in \operatorname{Dst} f: (a \not\asymp \langle f^{-1} \rangle x \Rightarrow b \not\asymp \langle f^{-1} \rangle x) \Rightarrow \forall x \in \mathfrak{A}: (x \not\asymp \langle f \rangle a \Rightarrow x \not\asymp \langle f \rangle b) \Rightarrow \langle f \rangle a \subseteq \langle f \rangle b$ (used the theorem 19 in [3]). □

Theorem 15. Let f is a pointfree funcoid from a distributive lattice $\operatorname{Src} f$ with least element to a separable distributive lattice $\operatorname{Dst} f$ with least element. Then $\langle f \rangle (i \cup^{\operatorname{Src} f} j) = \langle f \rangle i \cup^{\operatorname{Dst} f} \langle f \rangle j$ for every $i, j \in \operatorname{Src} f$. [TODO: In the book this theorem is strenghtened for starrish semilattices instead of distributive lattices.]

Proof.

$$\begin{aligned}
\star \langle f \rangle (i \cup^{\operatorname{Src} f} j) &= \\
\{ y \in \mathfrak{A} \mid y \cap^{\operatorname{Dst} f} \langle f \rangle (i \cup^{\operatorname{Src} f} j) \neq 0^{\operatorname{Dst} f} \} &= \\
\{ y \in \mathfrak{A} \mid (i \cup^{\operatorname{Src} f} j) \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y \neq 0^{\operatorname{Src} f} \} &= \\
\{ y \in \mathfrak{A} \mid (i \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y) \cup^{\operatorname{Src} f} (j \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y) \neq 0^{\operatorname{Src} f} \} &= \\
\{ y \in \mathfrak{A} \mid i \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y \neq 0^{\operatorname{Src} f} \vee j \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y \neq 0^{\operatorname{Src} f} \} &= \\
\{ y \in \mathfrak{A} \mid y \cap^{\operatorname{Dst} f} \langle f \rangle i \neq 0^{\operatorname{Dst} f} \vee y \cap^{\operatorname{Dst} f} \langle f \rangle j \neq 0^{\operatorname{Dst} f} \} &= \\
\{ y \in \mathfrak{A} \mid (y \cap^{\operatorname{Dst} f} \langle f \rangle i) \cup^{\operatorname{Dst} f} (y \cap^{\operatorname{Dst} f} \langle f \rangle j) \neq 0^{\operatorname{Dst} f} \} &= \\
\{ y \in \mathfrak{A} \mid y \cap^{\operatorname{Dst} f} (\langle f \rangle i \cup^{\operatorname{Dst} f} \langle f \rangle j) \neq 0^{\operatorname{Dst} f} \} &= \\
\star (\langle f \rangle i \cup^{\operatorname{Dst} f} \langle f \rangle j). &
\end{aligned}$$

Thus $\langle f \rangle (i \cup^{\operatorname{Src} f} j) = \langle f \rangle i \cup^{\operatorname{Src} f} \langle f \rangle j$ by separability. □

Proposition 16. Let f is a pointfree funcoid. Then: [TODO: This theorem is also strenghtened in the book.]

1. $k \, [f] \, i \cup j \Leftrightarrow k \, [f] \, i \vee k \, [f] \, k$ for every $i, j \in \operatorname{Dst} f$, $k \in \operatorname{Src} f$ if $\operatorname{Dst} f$ is a distributive lattice with least element.
2. $i \cup j \, [f] \, k \Leftrightarrow i \, [f] \, k \vee j \, [f] \, k$ for every $i, j \in \operatorname{Src} f$, $k \in \operatorname{Dst} f$ if $\operatorname{Src} f$ is a distributive lattice with least element.

Proof. 1. $k \, [f] \, i \cup^{\operatorname{Dst} f} j \Leftrightarrow (i \cup j) \cap^{\operatorname{Dst} f} \langle f \rangle k \neq 0^{\operatorname{Dst} f} \Leftrightarrow (i \cap^{\operatorname{Dst} f} \langle f \rangle k) \cup (j \cap^{\operatorname{Dst} f} \langle f \rangle k) \neq 0^{\operatorname{Dst} f} \Leftrightarrow i \cap^{\operatorname{Dst} f} \langle f \rangle k \neq 0^{\operatorname{Dst} f} \vee j \cap^{\operatorname{Dst} f} \langle f \rangle k \neq 0^{\operatorname{Dst} f} \Leftrightarrow k \, [f] \, i \vee k \, [f] \, j$.

2. Similar. □

3.2 Composition of pointfree funcoids

Definition 17. *Composition* of pointfree funcoids is defined by the formula

$$(\mathfrak{B}; \mathfrak{C}; \alpha_2; \beta_2) \circ (\mathfrak{A}; \mathfrak{B}; \alpha_1; \beta_1) = (\mathfrak{A}; \mathfrak{C}; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2).$$

Definition 18. I will call funcoids f and g *composable* when $\operatorname{Dst} f = \operatorname{Src} g$.

Proposition 19. If f, g are pointfree funcoids and $\operatorname{Dst} f = \operatorname{Src} g$ then $g \circ f$ is pointfree funcoid.

Proof. Let $f = (\mathfrak{A}; \mathfrak{B}; \alpha_1; \beta_1)$, $g = (\mathfrak{B}; \mathfrak{C}; \alpha_2; \beta_2)$. For every $x, y \in \mathfrak{A}$ we have

$$y \not\asymp^{\mathfrak{C}} (\alpha_2 \circ \alpha_1) x \Leftrightarrow y \not\asymp^{\mathfrak{C}} \alpha_2 \alpha_1 x \Leftrightarrow \alpha_1 x \not\asymp^{\mathfrak{B}} \beta_2 y \Leftrightarrow x \not\asymp^{\mathfrak{A}} \beta_1 \beta_2 y \Leftrightarrow x \not\asymp^{\mathfrak{A}} (\beta_1 \circ \beta_2) y.$$

So $(\mathfrak{A}; \mathfrak{C}; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2)$ is a pointfree funcoid. □

Obvious 20. $\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle$ for every composable pointfree funcoids f and g.

Theorem 21. $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$ for every composable pointfree funcoids f and g.

Proof.

$$\langle (g\circ f)^{-1}\rangle = \langle f^{-1}\rangle \circ \langle g^{-1}\rangle = \langle f^{-1}\circ g^{-1}\rangle;$$
$$\langle ((g\circ f)^{-1})^{-1}\rangle = \langle g\circ f\rangle = \langle (f^{-1}\circ g^{-1})^{-1}\rangle.$$

□

Proposition 22. $(h\circ g)\circ f = h\circ (g\circ f)$ for every composable pointfree funcoids f, g, h.

Proof.
$\langle (h\circ g)\circ f\rangle = \langle h\circ g\rangle \circ \langle f\rangle = \langle h\rangle \circ \langle g\rangle \circ \langle f\rangle = \langle h\rangle \circ \langle g\circ f\rangle = \langle h\circ (g\circ f)\rangle$.
$\langle ((h\circ g)\circ f)^{-1}\rangle = \langle f^{-1}\circ (h\circ g)^{-1}\rangle = \langle f^{-1}\circ g^{-1}\circ h^{-1}\rangle = \langle (g\circ f)^{-1}\circ h^{-1}\rangle = \langle (h\circ (g\circ f))^{-1}\rangle$. □

3.3 Pointfree funcoid as continuation

Proposition 23. Let f is a pointfree funcoid. Then for every $x\in \operatorname{Src} f$, $y\in \operatorname{Dst} f$ we have

1. If $(\operatorname{Src} f;\mathfrak{Z})$ is a filtrator with separable core then $x\,[f]\,y \Leftrightarrow \forall X\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})} x: X\,[f]\,y$.
2. If $(\operatorname{Dst} f;\mathfrak{Z})$ is a filtrator with separable core then $x\,[f]\,y \Leftrightarrow \forall Y\in \operatorname{up}^{(\operatorname{Dst} f;\mathfrak{Z})} x: x\,[f]\,Y$.

Proof. We will prove only the second because the first is similar.
$x\,[f]\,y \Leftrightarrow y \not\asymp^{\operatorname{Dst} f} \langle f\rangle x \Leftrightarrow \forall Y\in \operatorname{up} y: Y \not\asymp^{\operatorname{Dst} f} \langle f\rangle x \Leftrightarrow \forall Y\in \operatorname{up} y: x\,[f]\,Y$. □

Corollary 24. Let f is a pointfree funcoid and $(\operatorname{Src} f;\mathfrak{Z}_0)$, $(\operatorname{Dst} f;\mathfrak{Z}_1)$ are filtrators with separable core. Then

$$x\,[f]\,y \Leftrightarrow \forall X\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)} x, Y\in \operatorname{up}^{(\operatorname{Dst} f;\mathfrak{Z}_1)} y: X\,[f]\,Y.$$

Proof. Apply the proposition twice. □

Theorem 25. Let f be a pointfree funcoid. Let $(\operatorname{Src} f;\mathfrak{Z}_0)$ is a finitely meet-closed filtrator with separable core and $(\operatorname{Dst} f;\mathfrak{Z}_1)$ is a primary filtrator over a distributive lattice.

$$\langle f\rangle x = \bigcap^{\operatorname{Dst} f} \langle\langle f\rangle\rangle \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})} x.$$

Proof. By the previous proposition for every $y\in \operatorname{Dst} f$:

$$y \not\asymp^{\operatorname{Dst} f} \langle f\rangle x \Leftrightarrow x\,[f]\,y \Leftrightarrow \forall X\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)} x: X\,[f]\,y \Leftrightarrow \forall X\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)} x: y \not\asymp^{\operatorname{Dst} f} \langle f\rangle X.$$

Let's denote $W=\left\{ y\cap^{\operatorname{Dst} f} \langle f\rangle X \mid X\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)} x\right\}$. We will prove that W is a generalized filter base over $\mathfrak{Z}_1$. To prove this enough to show that $V=\left\{ \langle f\rangle X \mid X\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)} x\right\}$ is a generalized filter base.

Let $\mathcal{P},\mathcal{Q}\in V$. Then $\mathcal{P}=\langle f\rangle A$, $\mathcal{Q}=\langle f\rangle B$ where $A,B\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)} x$; $A\cap^{\mathfrak{Z}_0} B\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)} x$ (used the fact that it is a finitely meet-closed and the theorem 29 in [3]) and $\mathcal{R}\subseteq \mathcal{P}\cap^{\operatorname{Dst} f}\mathcal{Q}$ for $\mathcal{R}=\langle f\rangle (A\cap^{\mathfrak{Z}_0} B)\in V$. So V is a generalized filter base and thus W is a generalized filter base.

$0^{\operatorname{Dst} f}\notin W \Leftrightarrow \bigcap^{\operatorname{Dst} f} W \neq 0^{\operatorname{Dst} f}$ by the properties of generalized filter bases. That is

$$\forall X\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)} x: y\cap^{\operatorname{Dst} f} \langle f\rangle X\neq 0^{\operatorname{Dst} f} \Leftrightarrow y\cap^{\operatorname{Dst} f} \bigcap^{\operatorname{Dst} f} \langle\langle f\rangle\rangle \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)} x\neq 0^{\operatorname{Dst} f}.$$

Comparing with the above, $y\cap^{\operatorname{Dst} f} \langle f\rangle x\neq 0^{\operatorname{Dst} f} \Leftrightarrow y\cap^{\operatorname{Dst} f} \bigcap^{\operatorname{Dst} f} \langle\langle f\rangle\rangle \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)} x\neq 0^{\operatorname{Dst} f}$. So $\langle f\rangle x = \bigcap^{\operatorname{Dst} f}\langle\langle f\rangle\rangle \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)} x$ because $\operatorname{Dst} f$ is separable. □

Theorem 26. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices.

1. A function $\alpha\in \mathfrak{B}^{\mathfrak{Z}_0}$ conforming to the formulas (for every $I,J\in \mathfrak{Z}_0$)

$$\alpha 0^{\mathfrak{Z}_0} = 0^{\mathfrak{B}},\quad \alpha (I\cup^{\mathfrak{Z}_0} J) = \alpha I\cup^{\mathfrak{B}} \alpha J$$

can be continued to the function $\langle f\rangle$ for a unique $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$;

$$\langle f\rangle\mathcal{X}=\bigcap{}^{\mathfrak{B}}\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X} \tag{1}$$

for every $\mathcal{X}\in\mathfrak{A}$.

2. A relation $\delta\in\mathscr{P}(\mathfrak{Z}_0\times\mathfrak{Z}_1)$ conforming to the formulas (for every I, J, $K\in\mathfrak{Z}_0$ and I', J', $K'\in\mathfrak{Z}_1$)

$$\begin{array}{ll}\neg(0^{\mathfrak{Z}_0}\,\delta\,I), & I\cup^{\mathfrak{Z}_0}J\,\delta\,K'\Leftrightarrow I\,\delta\,K'\vee J\,\delta\,K',\\ \neg(I'\,\delta\,0^{\mathfrak{Z}_1}), & K\,\delta\,I'\cup^{\mathfrak{Z}_1}J'\Leftrightarrow K\,\delta\,I'\vee K\,\delta\,J'\end{array} \tag{2}$$

can be continued to the relation $[f]$ for a unique $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$;

$$\mathcal{X}\,[f]\,\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X},Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}\mathcal{Y}\colon X\,\delta\,Y \tag{3}$$

for every $\mathcal{X}\in\mathfrak{A}$, $\mathcal{Y}\in\mathfrak{B}$.

Proof. Existence of no more than one such pointfree funcoids and formulas (1) and (3) follow from two previous theorems.

2. $\{Y\in\mathfrak{Z}_1\mid X\,\delta\,Y\}$ is obviously a free star for every $X\in\mathfrak{Z}_0$. By properties of filters on boolean lattices, there exist a unique filter object αX such that $\partial(\alpha X)=\{Y\in\mathfrak{Z}_1\mid X\,\delta\,Y\}$ for every $X\in\mathfrak{Z}_0$. Thus $\alpha\in\mathfrak{B}^{\mathfrak{Z}_0}$. Similarly it can be defined $\beta\in\mathfrak{A}^{\mathfrak{Z}_1}$ by the formula $\partial(\beta X)=\{X\in\mathfrak{Z}_0\mid X\,\delta\,Y\}$. Let's continue the functions α and β to $\alpha'\in\mathfrak{B}^{\mathfrak{A}}$ and $\beta'\in\mathfrak{A}^{\mathfrak{B}}$ by the formulas

$$\alpha'\mathcal{X}=\bigcap{}^{\mathfrak{B}}\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}\quad\text{and}\quad\beta'\mathcal{X}=\bigcap{}^{\mathfrak{A}}\langle\beta\rangle\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}\mathcal{X}$$

and δ to $\delta'\in\mathscr{P}(\mathfrak{A}\times\mathfrak{B})$ by the formula

$$\mathcal{X}\,\delta'\,\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X},Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}\mathcal{Y}\colon X\,\delta\,Y.$$

$\mathcal{Y}\cap^{\mathfrak{B}}\alpha'\mathcal{X}\neq0^{\mathfrak{B}}\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{B}}\bigcap{}^{\mathfrak{B}}\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}\neq0^{\mathfrak{B}}\Leftrightarrow\bigcap{}^{\mathfrak{B}}\langle\mathcal{Y}\cap^{\mathfrak{B}}\rangle\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}\neq0^{\mathfrak{B}}$. Let's prove that

$$W=\langle\mathcal{Y}\cap^{\mathfrak{B}}\rangle\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$$

is a generalized filter base: To prove it is enough to show that $\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$ is a generalized filter base.

If $\mathcal{A},\mathcal{B}\in\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$ then exist $X_1,X_2\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$ such that $\mathcal{A}=\alpha X_1$ and $\mathcal{B}=\alpha X_2$.

Then $\alpha(X_1\cap^{\mathfrak{Z}_0}X_2)\in\langle\alpha\rangle\mathrm{up}\,\mathcal{X}$. So $\langle\alpha\rangle\mathrm{up}\,\mathcal{X}$ is a generalized filter base and thus W is a generalized filter base.

By properties of generalized filter bases, $\bigcap{}^{\mathfrak{B}}\langle\mathcal{Y}\cap^{\mathfrak{B}}\rangle\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}\neq0^{\mathfrak{B}}$ is equivalent to

$$\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}\colon\mathcal{Y}\cap^{\mathfrak{B}}\alpha X\neq0^{\mathfrak{B}},$$

what is equivalent to $\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$, $Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}\mathcal{Y}$: $Y\cap^{\mathfrak{B}}\alpha X\neq0^{\mathfrak{B}}\Leftrightarrow\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$, $Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}\mathcal{Y}$: $Y\in\partial(\alpha X)\Leftrightarrow\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$, $Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}\mathcal{Y}$: $X\ \delta\ Y$. Combining the equivalencies we get $\mathcal{Y}\cap^{\mathfrak{B}}\alpha'\mathcal{X}\neq0\Leftrightarrow X\ \delta'\ Y$. Analogously $\mathcal{X}\cap^{\mathfrak{A}}\beta'\mathcal{Y}\neq0^{\mathfrak{A}}\Leftrightarrow X\ \delta'\ Y$. So $\mathcal{Y}\cap^{\mathfrak{B}}\alpha'\mathcal{X}\neq0^{\mathfrak{B}}\Leftrightarrow\mathcal{X}\cap^{\mathfrak{A}}\beta'\mathcal{Y}\neq0^{\mathfrak{A}}$, that is $(\mathfrak{A};\mathfrak{B};\alpha';\beta')$ is a pointfree funcoid. From the formula $\mathcal{Y}\cap^{\mathfrak{B}}\alpha'\mathcal{X}\neq\emptyset\Leftrightarrow X\,\delta'\,Y$ follows that $[(\mathfrak{A};\mathfrak{B};\alpha';\beta')]$ is a continuation of δ.

1. Let define the relation $\delta\in\mathscr{P}(\mathfrak{Z}_0\times\mathfrak{Z}_1)$ by the formula $X\,\delta\,Y\Leftrightarrow Y\cap^{\mathfrak{B}}\alpha X\neq0^{\mathfrak{B}}$.

That $\neg(0^{\mathfrak{Z}_0}\,\delta\,I')$ and $\neg(I\,\delta\,0^{\mathfrak{Z}_1})$ is obvious. We have $K\,\delta\,I'\cup^{\mathfrak{Z}_1}J'\Leftrightarrow(I'\cup^{\mathfrak{Z}_1}J')\cap^{\mathfrak{B}}\alpha K\neq0^{\mathfrak{B}}\Leftrightarrow(I'\cup^{\mathfrak{B}}J')\cap^{\mathfrak{B}}\alpha K\neq0^{\mathfrak{B}}\Leftrightarrow(I'\cap^{\mathfrak{B}}\alpha K)\cup^{\mathfrak{B}}(J'\cap^{\mathfrak{B}}\alpha K)\neq0^{\mathfrak{B}}\Leftrightarrow I'\cap^{\mathfrak{B}}\alpha K\neq0^{\mathfrak{B}}\vee J'\cap^{\mathfrak{B}}\alpha K\neq0^{\mathfrak{B}}\Leftrightarrow K\ \delta\ I'\vee K\ \delta\ J'$ and $I\cup^{\mathfrak{Z}_0}J\ \delta\ K'\Leftrightarrow K'\cap^{\mathfrak{B}}\alpha(I\cup^{\mathfrak{Z}_0}J)\neq0^{\mathfrak{B}}\Leftrightarrow K'\cap^{\mathfrak{B}}(\alpha I\cup^{\mathfrak{B}}\alpha J)\neq0^{\mathfrak{B}}\Leftrightarrow(K'\cap^{\mathfrak{B}}\alpha I)\cup^{\mathfrak{B}}(K'\cap^{\mathfrak{B}}\alpha J)\neq0^{\mathfrak{B}}\Leftrightarrow K'\cap^{\mathfrak{B}}\alpha I\neq0^{\mathfrak{B}}\vee K'\cap^{\mathfrak{B}}\alpha J\neq0^{\mathfrak{B}}\Leftrightarrow I\,\delta\,K'\vee J\,\delta\,K'$.

That is the formulas (2) are true.

Accordingly the above δ can be continued to the relation $[f]$ for some $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$.

$\forall X\in\mathfrak{Z}_0,Y\in\mathfrak{Z}_1\colon(Y\cap^{\mathfrak{B}}\langle f\rangle X\neq0\Leftrightarrow X\,[f]\,Y\Leftrightarrow Y\cap^{\mathfrak{B}}\alpha X\neq0^{\mathfrak{B}})$, consequently $\forall X\in\mathfrak{Z}_0\colon\alpha X=\langle f\rangle X$ because our filtrator is with separable core. So $\langle f\rangle$ is a continuation of α. □

Proposition 27. Let $(\mathrm{Src}\,f;\mathfrak{Z}_0)$ is a primary filtrator over a bounded distributive lattice element and $(\mathrm{Dst}\,f;\mathfrak{Z}_1)$ is a primary filtrator over a distributive lattice. If S is a generalized filter base on $\mathrm{Src}\,f$ then $\langle f\rangle\bigcap{}^{\mathrm{Src}\,f}S=\bigcap{}^{\mathrm{Dst}\,f}\langle\langle f\rangle\rangle S$ for every pointfree funcoid f.

Proof. $(\mathrm{Src}\, f; \mathfrak{Z}_0)$ is a finitely meet-closed filtrator by theorem 29 in [3] and with separable core by theorem 37 and corollary 10 in [3]; thus we can apply theorem 25.

$\langle f \rangle \bigcap^{\mathrm{Src}\, f} S \subseteq \langle f \rangle X$ for every $X \in S$ and thus $\langle f \rangle \bigcap^{\mathrm{Src}\, f} S \subseteq \bigcap^{\mathrm{Dst}\, f} \langle\langle f \rangle\rangle S$.

Taking into account properties of generalized filter bases:

$$\begin{aligned} \langle f \rangle \bigcap^{\mathrm{Src}\, f} S &= \\ \bigcap^{\mathrm{Dst}\, f} \langle\langle f \rangle\rangle \mathrm{up} \bigcap S &= \\ \bigcap^{\mathrm{Dst}\, f} \{\langle\langle f \rangle\rangle X \mid \exists \mathcal{P} \in S: X \in \mathrm{up}\, \mathcal{P}\} &= \\ \bigcap^{\mathrm{Dst}\, f} \{\langle f \rangle X \mid \exists \mathcal{P} \in S: X \in \mathrm{up}\, \mathcal{P}\} &\supseteq \\ \bigcap^{\mathrm{Dst}\, f} \{\langle f \rangle \mathcal{P} \mid \mathcal{P} \in S\} &= \\ \bigcap^{\mathrm{Dst}\, f} \langle\langle f \rangle\rangle S. & \end{aligned}$$

□

3.4 The preorder of pointfree funcoids

The *preorder of pointfree funcoids* is defined by the formula $f \subseteq g \Leftrightarrow [f] \subseteq [g]$ for every pointfree funcoids f and g.

Remark 28. It is enough to define preorder of pointfree funcoids on every set $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ where $\mathfrak{A}$ and $\mathfrak{B}$ are posets. We do not need to compare pointfree funcoids with different sources or destinations.

Theorem 29. If $\mathfrak{A}$ and $\mathfrak{B}$ are separable posets then $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is a poset.

Proof. From the theorem 12. □

Theorem 30. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices. Then for $R \in \mathscr{P}\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $X \in \mathfrak{Z}_0$, $Y \in \mathfrak{Z}_1$ we have:

1. $X \left[\bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R\right] Y \Leftrightarrow \exists f \in R: X [f] Y$;
2. $\left\langle \bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R \right\rangle X = \bigcup^{\mathfrak{B}} \{\langle f \rangle X \mid f \in R\}$.

Proof.

2. $\alpha X \overset{\mathrm{def}}{=} \bigcup^{\mathfrak{B}} \{\langle f \rangle X \mid f \in R\}$ (by corollary 8 in [3] all joins on $\mathfrak{B}$ exist). We have $\alpha 0^{\mathfrak{A}} = 0^{\mathfrak{B}}$;

$$\begin{aligned} \alpha(I \cup^{\mathfrak{Z}_0} J) &= \bigcup^{\mathfrak{B}} \{\langle f \rangle (I \cup^{\mathfrak{Z}_0} J) \mid f \in R\} \\ &= \bigcup^{\mathfrak{B}} \{\langle f \rangle (I \cup^{\mathfrak{A}} J) \mid f \in R\} \\ &= \bigcup^{\mathfrak{B}} \{\langle f \rangle I \cup^{\mathfrak{B}} \langle f \rangle J \mid f \in R\} \\ &= \bigcup^{\mathfrak{B}} \{\langle f \rangle I \mid f \in R\} \cup^{\mathfrak{B}} \bigcup^{\mathfrak{B}} \{\langle f \rangle J \mid f \in R\} \\ &= \alpha I \cup^{\mathfrak{B}} \alpha J \end{aligned}$$

(used the theorem 15). By the theorem 26 the function α can be continued to $\langle h \rangle$ for a $h \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$. Obviously

$$\forall f \in R: h \supseteq f. \tag{4}$$

And h is the least element of $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ for which holds the condition (4). So $h = \bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R$.

1. $X \left[\bigcup^{\mathsf{FCD}} R\right] Y \Leftrightarrow Y \cap^{\mathfrak{B}} \left\langle \bigcup^{\mathsf{FCD}} R \right\rangle X \neq 0^{\mathfrak{B}} \Leftrightarrow Y \cap^{\mathfrak{B}} \bigcup^{\mathfrak{B}} \{\langle f \rangle X \mid f \in R\} \neq 0^{\mathfrak{B}} \Leftrightarrow \exists f \in R: Y \cap^{\mathfrak{B}} \langle f \rangle X \neq 0^{\mathfrak{B}} \Leftrightarrow \exists f \in R: X [f] Y$ (used the theorem 40 in [3]). □

Corollary 31. If $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices then $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is a complete lattice.

Proof. $\mathfrak{A}$ and $\mathfrak{B}$ are separable accordingly obvious 20 in [3].

Then apply [1] taking in account the theorem 12. $\square$

Theorem 32. Let $\mathfrak{A}$ and $\mathfrak{B}$ are starrish join-semilattices. Then:

1. $\left\langle f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g \right\rangle x = \langle f \rangle x \cup^{\mathfrak{B}} \langle g \rangle x$ for every $x \in \mathfrak{A}$;
2. $\left[f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g \right] = [f] \cup [g]$.

Proof.

1. Let $\alpha \mathcal{X} \overset{\text{def}}{=} \langle f \rangle x \cup^{\mathfrak{B}} \langle g \rangle x$; $\beta \mathcal{Y} \overset{\text{def}}{=} \langle f^{-1} \rangle y \cup^{\mathfrak{A}} \langle g^{-1} \rangle y$ for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$. Then

$$\begin{aligned} y \not\asymp^{\mathfrak{B}} \alpha x &\Leftrightarrow y \not\asymp^{\mathfrak{B}} \langle f \rangle x \vee y \not\asymp^{\mathfrak{B}} \langle g \rangle x \\ &\Leftrightarrow x \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle y \vee x \not\asymp^{\mathfrak{A}} \langle g^{-1} \rangle y \\ &\Leftrightarrow x \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle y \cup^{\mathfrak{A}} \langle g^{-1} \rangle y \\ &\Leftrightarrow x \not\asymp^{\mathfrak{A}} \beta y. \end{aligned}$$

So $h = (\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$ is a pointfree funcoid. Obviously $h \supseteq f$ and $h \supseteq g$. If $p \supseteq f$ and $p \supseteq g$ for some $p \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ then $\langle p \rangle x \supseteq \langle f \rangle x \cup^{\mathfrak{B}} \langle g \rangle x = \langle h \rangle x$ that is $p \supseteq h$. So $f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g = h$.

2. $x \left[f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g \right] y \Leftrightarrow y \not\asymp^{\mathfrak{B}} \left\langle f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g \right\rangle x \Leftrightarrow y \not\asymp^{\mathfrak{B}} \langle f \rangle x \cup^{\mathfrak{B}} \langle g \rangle x \Leftrightarrow y \not\asymp^{\mathfrak{B}} \langle f \rangle x \vee y \not\asymp^{\mathfrak{B}} \langle g \rangle x \Leftrightarrow x [f] y \vee x [g] y$ for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$. $\square$

Theorem 33. Let f is a pointfree funcoid from a separable poset $\mathfrak{A}$ to a separable poset $\mathfrak{B}$. If $\langle f \rangle$ is an injection, then $\langle f \rangle$ is an order embedding $\mathfrak{A} \rightarrow \mathfrak{B}$.

Proof. Suppose $x \supseteq y$ but $\langle f \rangle x \not\supseteq \langle f \rangle y$.

Then by separability of $\mathfrak{B}$ there exist $z \not\asymp \langle f \rangle y$ such that $z \asymp \langle f \rangle x$.

Thus $\langle f^{-1} \rangle z \asymp x$ and $\langle f^{-1} \rangle z \not\asymp y$ what is impossible for $x \supseteq y$. $\square$

Corollary 34. Let f is a pointfree funcoid from a separable poset $\mathfrak{A}$ to a separable poset $\mathfrak{B}$. If $\langle f \rangle$ is a bijection $\mathfrak{A} \rightarrow \mathfrak{B}$, then $\langle f \rangle$ is an order isomorphism $\mathfrak{A} \rightarrow \mathfrak{B}$.

3.5 More on composition of pointfree funcoids

Proposition 35. $[g \circ f] = [g] \circ \langle f \rangle = \langle g^{-1} \rangle^{-1} \circ [f]$ for every composable pointfree funcoids f and g.

Proof. $x [g \circ f] y \Leftrightarrow y \not\asymp^{\operatorname{Dst} g} \langle g \circ f \rangle x \Leftrightarrow y \not\asymp^{\operatorname{Dst} g} \langle g \rangle \langle f \rangle x \Leftrightarrow \langle f \rangle x [g] y \Leftrightarrow x([g] \circ \langle f \rangle) y$ for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$. Thus $[g \circ f] = [g] \circ \langle f \rangle$. $[g \circ f] = [(f^{-1} \circ g^{-1})^{-1}] = [f^{-1} \circ g^{-1}]^{-1} = ([f^{-1}] \circ \langle g^{-1} \rangle)^{-1} = \langle g^{-1} \rangle^{-1} \circ [f]$. $\square$

Theorem 36. Let f and g are pointfree funcoids and $\mathfrak{A} = \operatorname{Dst} f = \operatorname{Src} g$ is an atomic poset. Then for every $\mathcal{X} \in \operatorname{Src} f$ and $\mathcal{Z} \in \operatorname{Dst} g$

$$\mathcal{X} [g \circ f] \mathcal{Z} \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{A}} : (\mathcal{X} [f] y \wedge y [g] \mathcal{Z}).$$

Proof.

$$\begin{aligned} \exists y \in \operatorname{atoms}^{\mathfrak{A}} : (\mathcal{X} [f] y \wedge y [g] \mathcal{Z}) &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{A}} : (\mathcal{Z} \not\asymp^{\operatorname{Dst} g} \langle g \rangle y \wedge y \not\asymp^{\mathfrak{A}} \langle f \rangle \mathcal{X}) \\ &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{A}} : (\mathcal{Z} \not\asymp^{\operatorname{Dst} g} \langle g \rangle y \wedge y \subseteq \langle f \rangle \mathcal{X}) \\ &\Rightarrow \mathcal{Z} \not\asymp^{\operatorname{Dst} g} \langle g \rangle \langle f \rangle \mathcal{X} \\ &\Leftrightarrow \mathcal{X} [g \circ f] \mathcal{Z}. \end{aligned}$$

Reversely, if $\mathcal{X} [g \circ f] \mathcal{Z}$ then $\langle f \rangle \mathcal{X} [g] \mathcal{Z}$, consequently exists $y \in \operatorname{atoms}^{\mathfrak{A}} \langle f \rangle \mathcal{X}$ such that $y [g] \mathcal{Z}$; we have $\mathcal{X} [f] y$. $\square$

Theorem 37. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are posets and $\mathfrak{B}$ is atomic. Then:

1. $f \circ (g \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} h) = f \circ g \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{C})} f \circ h$ for $g, h \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $f \in \mathsf{FCD}(\mathfrak{B}; \mathfrak{C})$.

2. $(g \cup^{\mathsf{FCD}(\mathfrak{B};\mathfrak{C})} h) \circ f = g \circ f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{C})} h \circ f$ for $f \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $g, h \in \mathsf{FCD}(\mathfrak{B};\mathfrak{C})$.

Proof. I will prove only the first equality because the other is analogous.

For every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{C}$

$$\begin{aligned}
\mathcal{X} \left[f \circ (g \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} h) \right] \mathcal{Z} &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{B}}: (\mathcal{X} \left[g \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} h \right] y \wedge y\, [f]\, \mathcal{Z}) \\
&\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{B}}: ((\mathcal{X}\, [g]\, y \vee \mathcal{X}\, [h]\, y) \wedge y\, [f]\, \mathcal{Z}) \\
&\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{B}}: ((\mathcal{X}\, [g]\, y \wedge y\, [f]\, \mathcal{Z}) \vee (\mathcal{X}\, [h]\, y \wedge y\, [f]\, \mathcal{Z})) \\
&\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{B}}: (\mathcal{X}\, [g]\, y \wedge y\, [f]\, \mathcal{Z}) \vee \exists y \in \operatorname{atoms}^{\mathfrak{B}}: (\mathcal{X}\, [h]\, y \wedge y\, [f]\, \mathcal{Z}) \\
&\Leftrightarrow \mathcal{X}\, [f \circ g]\, \mathcal{Z} \vee \mathcal{X}\, [f \circ h]\, \mathcal{Z} \\
&\Leftrightarrow \mathcal{X} \left[f \circ g \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{C})} f \circ h \right] \mathcal{Z}.
\end{aligned}$$

□

3.6 Domain and range of a pointfree funcoid

Definition 38. Let $\mathfrak{A}$ is a poset. The *identity pointfree funcoid* $I^{\mathsf{FCD}(\mathfrak{A})} = (\mathfrak{A}; \mathfrak{A}; (=)|_{\mathfrak{A}}; (=)|_{\mathfrak{A}})$.

It is trivial that identity funcoid is really a pointfree funcoid.

Let now $\mathfrak{A}$ is a meet-semilattice.

Definition 39. Let $a \in \mathfrak{A}$. The *restricted identity pointfree funcoid* $I_a^{\mathsf{FCD}(\mathfrak{A})} = (\mathfrak{A}; \mathfrak{A}; a \cap^{\mathfrak{A}}; a \cap^{\mathfrak{A}})$.

Proposition 40. The restricted pointfree funcoid is a pointfree funcoid.

Proof. We need to prove that $(a \cap^{\mathfrak{A}} x) \not\asymp^{\mathfrak{A}} y \Leftrightarrow (a \cap^{\mathfrak{A}} y) \not\asymp^{\mathfrak{A}} x$ what is obvious. □

Obvious 41. $\left(I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} \right)^{-1} = I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}$.

Obvious 42. $x \left[I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} \right] y \Leftrightarrow a \not\asymp^{\mathfrak{A}} x \cap^{\mathfrak{A}} y$ for every $x, y \in \mathfrak{A}$.

Definition 43. I will define *restricting* of a pointfree funcoid f to an element $a \in \operatorname{Src} f$ by the formula $f|_a \overset{\text{def}}{=} f \circ I_a^{\mathsf{FCD}(\operatorname{Src} f)}$.

Definition 44. Let f is a pointfree funcoid whose source has greatest element 1.

Image of f will be defined by the formula $\operatorname{im} f = \langle f \rangle 1$.

Definition 45. *Domain* of a pointfree funcoid f is defined by the formula $\operatorname{dom} f = \operatorname{im} f^{-1}$ (when f has a poset with greatest element as its destination).

Proposition 46. $\langle f \rangle x = \langle f \rangle (x \cap^{\operatorname{Src} f} \operatorname{dom} f)$ for every pointfree funcoid f whose destination is a separable poset with greatest element and source is a meet-semilattice and $x \in \operatorname{Src} f$.

Proof. For every $y \in \operatorname{Dst} f$ we have $y \not\asymp^{\operatorname{Dst} f} \langle f \rangle (x \cap^{\operatorname{Src} f} \operatorname{dom} f) \neq 0^{\operatorname{Dst} f} \Leftrightarrow x \cap^{\operatorname{Src} f} \operatorname{dom} f \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y \neq 0^{\operatorname{Src} f} \Leftrightarrow x \cap^{\operatorname{Src} f} \operatorname{im} f^{-1} \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y \neq 0^{\operatorname{Src} f} \Leftrightarrow x \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y \neq 0^{\operatorname{Src} f} \Leftrightarrow y \not\asymp^{\operatorname{Dst} f} \langle f \rangle x$. Thus $\langle f \rangle x = \langle f \rangle (x \cap^{\operatorname{Src} f} \operatorname{dom} f)$ by separability of $\operatorname{Dst} f$. □

Proposition 47. $x \not\asymp^{\operatorname{Src} f} \operatorname{dom} f \Leftrightarrow (\langle f \rangle x$ is not least) for every pointfree funcoid f whose destination is a poset with greatest element and $x \in \operatorname{Src} f$.

Proof. $x \not\asymp^{\operatorname{Src} f} \operatorname{dom} f \Leftrightarrow x \not\asymp^{\operatorname{Src} f} \langle f^{-1} \rangle 1 \Leftrightarrow 1^{\operatorname{Dst} f} \not\asymp^{\operatorname{Dst} f} \langle f \rangle x \Leftrightarrow (\langle f \rangle x$ is not least). □

Corollary 48. $\operatorname{dom} f = \bigcup^{\operatorname{Src} f} \{ a \in \operatorname{atoms}^{\operatorname{Src} f} \mid \langle f \rangle a \neq 0^{\operatorname{Dst} f} \}$ for every pointfree funcoid f whose destination is a bounded poset and source is an atomistic meet-semilattice.

Proof. For every $a \in \operatorname{atoms}^{\operatorname{Src} f}$ we have $a \not\asymp^{\operatorname{Src} f} \operatorname{dom} f \Leftrightarrow a \not\asymp^{\operatorname{Src} f} \operatorname{im} f^{-1} \Leftrightarrow a \not\asymp^{\operatorname{Src} f} \langle f^{-1} \rangle 1^{\operatorname{Dst} f} \Leftrightarrow 1^{\operatorname{Dst} f} \not\asymp \langle f \rangle a \Leftrightarrow \langle f \rangle a \neq 0^{\operatorname{Dst} f}$. So

$$\operatorname{dom} f = \bigcup^{\operatorname{Src} f}\{a \in \operatorname{atoms}^{\operatorname{Src} f} \mid a \not\asymp^{\operatorname{Src} f} \operatorname{dom} f\} = \bigcup^{\operatorname{Src} f}\{a \in \operatorname{atoms}^{\operatorname{Src} f} \mid \langle f \rangle a \neq 0^{\operatorname{Dst} f}\}. \qquad \square$$

Proposition 49. $\operatorname{dom} f|_a^{\mathsf{FCD}(\operatorname{Src} f)} = a \sqcap^{\operatorname{Src} f} \operatorname{dom} f$ for every pointfree funcoid f and $a \in \operatorname{Src} f$ where $\operatorname{Src} f$ is a meet-semilattice and $\operatorname{Dst} f$ has greatest element.

Proof. $\operatorname{dom} f|_a^{\mathsf{FCD}(\operatorname{Src} f)} = \operatorname{im}\left(I_a^{\mathsf{FCD}(\operatorname{Src} f)} \circ f^{-1}\right) = \left\langle I_a^{\mathsf{FCD}(\operatorname{Src} f)} \right\rangle \langle f^{-1} \rangle 1^{\operatorname{Dst} f} = a \sqcap^{\operatorname{Src} f} \langle f^{-1} \rangle 1^{\operatorname{Dst} f} = a \sqcap^{\operatorname{Src} f} \operatorname{dom} f$. $\square$

Proposition 50. For every composable pointfree funcoids f and g where the posets $\operatorname{Src} f$ and $\operatorname{Dst} f = \operatorname{Src} g$ have greatest elements:

1. If $\operatorname{im} f \supseteq \operatorname{dom} g$ then $\operatorname{im}(g \circ f) = \operatorname{im} g$.
2. If $\operatorname{im} f \subseteq \operatorname{dom} g$ then $\operatorname{dom}(g \circ f) = \operatorname{dom} g$.

Proof.

1. $\operatorname{im}(g \circ f) = \langle g \circ f \rangle 1^{\operatorname{Src} f} = \langle g \rangle \langle f \rangle 1^{\operatorname{Src} f} = \langle g \rangle \operatorname{im} f = \langle g \rangle (\operatorname{im} f \sqcap^{\operatorname{Dst} f} \operatorname{dom} g) = \langle g \rangle \operatorname{dom} g = \langle g \rangle 1^{\operatorname{Src} g} = \operatorname{im} g$.
2. $\operatorname{dom}(g \circ f) = \operatorname{im}(f^{-1} \circ g^{-1})$ what by the proved is equal to $\operatorname{im} f^{-1}$ that is $\operatorname{dom} f$. $\square$

3.7 Category of pointfree funcoids

I will define the category $\mathrm{pf}\mathsf{FCD}$ of pointfree funcoids:

- The class of objects are small posets.
- The set of morphisms from $\mathfrak{A}$ to $\mathfrak{B}$ is $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.
- The composition is the composition of pointfree funcoids.
- Identity morphism for an object $\mathfrak{A}$ is $(\mathfrak{A}; \mathfrak{A}; (=)|_{\mathfrak{A}}; (=)|_{\mathfrak{A}})$.

To prove that it is really a category is trivial.

The *category of funcoid triples* is defined as follows:

- Objects are pairs $(\mathfrak{A}; \mathcal{A})$ where $\mathfrak{A}$ is a small poset and $\mathcal{A} \in \mathfrak{A}$.
- The morphisms from an object $(\mathfrak{A}; \mathcal{A})$ to an object $(\mathfrak{B}; \mathcal{B})$ are tuples $(f; \mathfrak{A}; \mathfrak{B}; \mathcal{A}; \mathcal{B})$ where $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $\operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$.
- The composition is defined by the formula $(g; \mathcal{B}; \mathcal{C}) \circ (f; \mathcal{A}; \mathcal{B}) = (g \circ f; \mathcal{A}; \mathcal{C})$.
- Identity morphism for an object $(\mathfrak{A}; \mathcal{A})$ is $I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}$.

To prove that it is really a category is trivial.

3.8 Specifying funcoids by functions or relations on atomic filter objects

Theorem 51. Let $\mathfrak{A}$ is an atomic poset and $(\mathfrak{B}; \mathfrak{Z}_1)$ is a primary filtrator over a boolean lattice. Then for every $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $\mathcal{X} \in \mathfrak{A}$ we have

$$\langle f \rangle \mathcal{X} = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}.$$

Proof. For every $Y \in \mathfrak{Z}_1$ we have

$$\begin{aligned} Y \not\asymp^{\mathfrak{B}} \langle f \rangle \mathcal{X} &\Leftrightarrow \mathcal{X} \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle Y \\ &\Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} : x \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle Y \\ &\Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} : Y \not\asymp^{\mathfrak{B}} \langle f \rangle x. \end{aligned}$$

Thus $\partial \langle f \rangle \mathcal{X} = \bigcup \langle \partial \rangle \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} = \partial \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$ (used the theorem 46 in [3]). Consequently $\langle f \rangle \mathcal{X} = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$ by the corollary 15 in [3]. $\square$

Proposition 52. Let f is a pointfree funcoid. Then for every $\mathcal{X} \in \operatorname{Src} f$ and $\mathcal{Y} \in \operatorname{Dst} f$

1. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms}^{\operatorname{Src} f} \mathcal{X} : x [f] \mathcal{Y}$ if $\operatorname{Src} f$ is an atomic poset.
2. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists y \in \operatorname{atoms}^{\operatorname{Dst} f} \mathcal{Y} : \mathcal{X} [f] y$ if $\operatorname{Dst} f$ is an atomic poset.

Proof. I will prove only the second as the first is similar.

If $\mathcal{X} [f] \mathcal{Y}$, then $\mathcal{Y} \not\asymp^{\operatorname{Dst} f} \langle f \rangle \mathcal{X}$, consequently exists $y \in \operatorname{atoms}^{\operatorname{Dst} f} \mathcal{Y}$ such that $y \not\asymp^{\operatorname{Dst} f} \langle f \rangle \mathcal{X}$, $\mathcal{X} [f] y$. The reverse is obvious. □

Corollary 53. If f is a pointfree funcoid with both source and destination being atomic posets, then for every $\mathcal{X} \in \operatorname{Src} f$ and $\mathcal{Y} \in \operatorname{Dst} f$

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms}^{\operatorname{Src} f} \mathcal{X}, y \in \operatorname{atoms}^{\operatorname{Dst} f} \mathcal{Y} : x [f] y.$$

Proof. Apply the theorem twice. □

Corollary 54. If $\mathfrak{A}$ is a separable atomic poset and $\mathfrak{B}$ is an atomistic poset then f is determined by the values of $\langle f \rangle X$ for $X \in \operatorname{atoms}^{\mathfrak{A}}$.

Proof. $y \not\asymp^{\operatorname{Dst} f} \langle f \rangle x \Leftrightarrow x^{\operatorname{Src} f} \not\asymp \langle f^{-1} \rangle y \Leftrightarrow \exists X \in \operatorname{atoms} x : X \not\asymp^{\operatorname{Src} f} \langle f^{-1} \rangle y \Leftrightarrow \exists X \in \operatorname{atoms} x : y \not\asymp^{\operatorname{Dst} f} \langle f \rangle X$.

Thus by atomisticity $\langle f \rangle$ is determined by $\langle f \rangle X$ for $X \in \operatorname{atoms} x$.

By separability of $\mathfrak{A}$ we infer that f can be restored from $\langle f \rangle$. □

Theorem 55. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices.

1. A function $\alpha \in \mathfrak{B}^{\operatorname{atoms}^{\mathfrak{A}}}$ such that (for every $a \in \operatorname{atoms}^{\mathfrak{A}}$)

$$\alpha a \subseteq \bigcap^{\mathfrak{B}} \left\langle \bigcup^{\mathfrak{B}} \circ \langle \alpha \rangle \circ \operatorname{atoms}^{\mathfrak{A}} \right\rangle \operatorname{up}^{(\mathfrak{A}; \mathfrak{Z}_0)} a \tag{5}$$

can be continued to the function $\langle f \rangle$ for a unique $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$;

$$\langle f \rangle \mathcal{X} = \bigcup^{\mathfrak{B}} \langle \alpha \rangle \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} \tag{6}$$

for every $\mathcal{X} \in \mathfrak{A}$.

2. A relation $\delta \in \mathscr{P}(\operatorname{atoms}^{\mathfrak{A}} \times \operatorname{atoms}^{\mathfrak{B}})$ such that (for every $a \in \operatorname{atoms}^{\mathfrak{A}}$, $b \in \operatorname{atoms}^{\mathfrak{B}}$)

$$\forall X \in \operatorname{up}^{(\mathfrak{A}; \mathfrak{Z}_0)} a, Y \in \operatorname{up}^{(\mathfrak{B}; \mathfrak{Z}_1)} b \, \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \, \delta \, y \Rightarrow a \, \delta \, b \tag{7}$$

can be continued to the relation $[f]$ for a unique $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$;

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}, y \in \operatorname{atoms}^{\mathfrak{B}} \mathcal{Y} : x \, \delta \, y \tag{8}$$

for every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{B}$.

Proof. Existence of no more than one such funcoids and formulas (6) and (8) follow from the theorem 51 and corollary 53 and the fact that our filtrators are with separable core.

1. Consider the function $\alpha' \in \mathfrak{B}^{\mathfrak{Z}_0}$ defined by the formula (for every $X \in \mathfrak{Z}_0$)

$$\alpha' X = \bigcup^{\mathfrak{B}} \langle \alpha \rangle \operatorname{atoms}^{\mathfrak{A}} X.$$

Obviously $\alpha' 0^{\mathfrak{Z}_0} = 0^{\mathfrak{B}}$. For every $I, J \in \mathfrak{Z}_0$

$$\begin{aligned}
\alpha'(I \cup J) &= \bigcup^{\mathfrak{B}} \langle \alpha' \rangle \operatorname{atoms}^{\mathfrak{B}} (I \cup J) \\
&= \bigcup^{\mathfrak{B}} \langle \alpha' \rangle (\operatorname{atoms}^{\mathfrak{B}} I \cup \operatorname{atoms}^{\mathfrak{B}} J) \\
&= \bigcup^{\mathfrak{B}} (\langle \alpha' \rangle \operatorname{atoms}^{\mathfrak{B}} I \cup \langle \alpha' \rangle \operatorname{atoms}^{\mathfrak{B}} J) \\
&= \bigcup^{\mathfrak{B}} \langle \alpha' \rangle \operatorname{atoms}^{\mathfrak{B}} I \cup^{\mathfrak{B}} \bigcup^{\mathfrak{B}} \langle \alpha' \rangle \operatorname{atoms}^{\mathfrak{B}} J. \\
&= \alpha' I \cup^{\mathfrak{B}} \alpha' J.
\end{aligned}$$

Let continue α' till a funcoid f (by the theorem 26): $\langle f\rangle\mathcal{X}=\bigcap^{\mathfrak{B}}\langle\alpha'\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$.

Let's prove the reverse of (5):

$$\begin{aligned}\bigcap^{\mathfrak{B}}\left\langle\bigcup^{\mathfrak{B}}\circ\langle\alpha\rangle\circ\mathrm{atoms}^{\mathfrak{A}}\right\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a &= \bigcap^{\mathfrak{B}}\left\langle\bigcup^{\mathfrak{B}}\circ\langle\alpha\rangle\right\rangle\langle\mathrm{atoms}^{\mathfrak{A}}\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a\\ &\subseteq \bigcap^{\mathfrak{B}}\left\langle\bigcup^{\mathfrak{B}}\circ\langle\alpha\rangle\right\rangle\{\{a\}\}\\ &= \bigcap^{\mathfrak{B}}\left\{\left(\bigcup^{\mathfrak{B}}\circ\langle\alpha\rangle\right)\{a\}\right\}\\ &= \bigcap^{\mathfrak{B}}\left\{\bigcup^{\mathfrak{B}}\langle\alpha\rangle\{a\}\right\}\\ &= \bigcap^{\mathfrak{B}}\left\{\bigcup^{\mathfrak{B}}\{\alpha a\}\right\}=\bigcap^{\mathfrak{B}}\{\alpha a\}=\alpha a.\end{aligned}$$

Finally,

$$\alpha a=\bigcap^{\mathfrak{B}}\left\langle\bigcup^{\mathfrak{B}}\circ\langle\alpha\rangle\circ\mathrm{atoms}^{\mathfrak{A}}\right\rangle\mathrm{up}\,a=\bigcap^{\mathfrak{B}}\langle\alpha'\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a=\langle f\rangle a,$$

so $\langle f\rangle$ is a continuation of α.

2. Consider the relation $\delta'\in\mathscr{P}(\mathfrak{Z}_0\times\mathfrak{Z}_1)$ defined by the formula (for every $X\in\mathfrak{Z}_0$, $Y\in\mathfrak{Z}_1$)

$$X\,\delta'\,Y\Leftrightarrow\exists x\in\mathrm{atoms}^{\mathfrak{A}}X,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y.$$

Obviously $\neg(X\,\delta'\,0^{\mathfrak{Z}_1})$ and $\neg(0^{\mathfrak{Z}_0}\,\delta'\,Y)$.

$$\begin{aligned}(I\cup J)\,\delta'\,Y &\Leftrightarrow \exists x\in\mathrm{atoms}^{\mathfrak{A}}(I\cup J),y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y\\ &\Leftrightarrow \exists x\in\mathrm{atoms}^{\mathfrak{A}}I\cup\mathrm{atoms}^{\mathfrak{A}}J,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y\\ &\Leftrightarrow \exists x\in\mathrm{atoms}^{\mathfrak{A}}I,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y\vee\exists x\in\mathrm{atoms}^{\mathfrak{A}}J,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y\\ &\Leftrightarrow I\,\delta'\,Y\vee J\,\delta'\,Y;\end{aligned}$$

similarly $X\,\delta'\,(I\cup J)\Leftrightarrow X\,\delta'\,I\vee X\,\delta'\,J$. Let's continue δ' till a funcoid f (by the theorem 26):

$$\mathcal{X}\,[f]\,\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X},Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}\mathcal{Y}:X\,\delta'\,Y$$

The reverse of (7) implication is trivial, so

$$\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a,Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b\,\exists x\in\mathrm{atoms}^{\mathfrak{A}}X,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y\Leftrightarrow a\,\delta\,b.$$

$\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a,Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b\,\exists x\in\mathrm{atoms}^{\mathfrak{A}}X,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y\Leftrightarrow\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a,Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b$: $X\,\delta'\,Y\Leftrightarrow a\,[f]\,b$.

So $a\,\delta\,b\Leftrightarrow a\,[f]\,b$, that is $[f]$ is a continuation of δ. □

One of uses of the previous theorem is the proof of the following theorem:

Theorem 56. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices. If $R\in\mathscr{P}\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $x\in\mathrm{atoms}^{\mathfrak{A}}$, $y\in\mathrm{atoms}^{\mathfrak{B}}$, then

1. $\left\langle\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R\right\rangle x=\bigcap^{\mathfrak{B}}\{\langle f\rangle x\mid f\in R\}$;
2. $x\left[\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R\right]y\Leftrightarrow\forall f\in R:x\,[f]\,y$.

Proof.

2. Let denote $x\,\delta\,y\Leftrightarrow\forall f\in R:x\,[f]\,y$.

$$\begin{gathered}\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a,Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b\,\exists x\in\mathrm{atoms}^{\mathfrak{A}}X,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,\delta\,y\Rightarrow\\ \forall f\in R,X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a,Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b\,\exists x\in\mathrm{atoms}^{\mathfrak{A}}X,y\in\mathrm{atoms}^{\mathfrak{B}}Y:x\,[f]\,y\Rightarrow\\ \forall f\in R,X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a,Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b:X\,[f]\,Y\Rightarrow\\ \forall f\in R:a\,[f]\,b\Leftrightarrow\\ a\,\delta\,b.\end{gathered}$$

So, by the theorem 55, δ can be continued till $[p]$ for some $p\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$.

For every $q\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ such that $\forall f\in R:q\subseteq f$ we have $x\,[q]\,y\Rightarrow\forall f\in R:x\,[f]\,y\Leftrightarrow x\,\delta\,y\Leftrightarrow x\,[p]\,y$, so $q\subseteq p$. Consequently $p=\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R$.

From this $x\left[\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R\right]y\Leftrightarrow\forall f\in R: x\,[f]\,y$.

1. From the former $y\in\operatorname{atoms}^{\mathfrak{B}}\langle\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R\rangle x\Leftrightarrow y\cap^{\mathfrak{B}}\langle\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R\rangle x\neq 0^{\mathfrak{B}}\Leftrightarrow\forall f\in R$: $y\cap^{\mathfrak{B}}\langle f\rangle x\neq 0^{\mathfrak{B}}\Leftrightarrow y\in\bigcap\langle\operatorname{atoms}^{\mathfrak{B}}\rangle\{\langle f\rangle x\mid f\in R\}\Leftrightarrow y\in\operatorname{atoms}^{\mathfrak{B}}\bigcap^{\mathfrak{B}}\{\langle f\rangle x\mid f\in R\}$ for every $y\in\operatorname{atoms}^{\mathfrak{B}}$.

$\mathfrak{B}$ is atomically separable by the corollary 17 in [3]. Thus

$$\langle\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R\rangle x=\bigcap^{\mathfrak{B}}\{\langle f\rangle x\mid f\in R\}.$$

□

Theorem 57. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are posets of filter objects over some boolean lattices, $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$, $g\in\mathsf{FCD}(\mathfrak{B};\mathfrak{C})$, $h\in\mathsf{FCD}(\mathfrak{A};\mathfrak{C})$. Then

$$g\circ f\not\asymp h\Leftrightarrow g\not\asymp h\circ f^{-1}.$$

Proof.

$$\begin{aligned}
g\circ f\not\asymp h &\Leftrightarrow\\
\exists a\in\operatorname{atoms}1^{\mathfrak{A}},c\in\operatorname{atoms}1^{\mathfrak{C}}: a\,[(g\circ f)\cap h]\,c &\Leftrightarrow\\
\exists a\in\operatorname{atoms}1^{\mathfrak{A}},c\in\operatorname{atoms}1^{\mathfrak{C}}: (a\,[g\circ f]\,c\wedge a\,[h]\,c) &\Leftrightarrow\\
\exists a\in\operatorname{atoms}1^{\mathfrak{A}},b\in\operatorname{atoms}1^{\mathfrak{B}},c\in\operatorname{atoms}1^{\mathfrak{C}}: (a\,[f]\,b\wedge b\,[g]\,c\wedge a\,[h]\,c) &\Leftrightarrow\\
\exists b\in\operatorname{atoms}1^{\mathfrak{B}},c\in\operatorname{atoms}1^{\mathfrak{C}}: (b\,[g]\,c\wedge b\,[h\circ f^{-1}]\,c) &\Leftrightarrow\\
\exists b\in\operatorname{atoms}1^{\mathfrak{B}},c\in\operatorname{atoms}1^{\mathfrak{C}}: b\,[g\cap(h\circ f^{-1})]\,c &\Leftrightarrow\\
g\not\asymp h\circ f^{-1}.
\end{aligned}$$

□

3.9 Direct product of elements

Definition 58. Let $\mathfrak{A}$ and $\mathfrak{B}$ are posets with least elements and $\mathcal{A}\in\mathfrak{A}$, $\mathcal{B}\in\mathfrak{B}$. *Funcoidal product* of $\mathcal{A},\mathcal{B}\in\mathfrak{A}$ is such a pointfree funcoid $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ that

$$\mathcal{X}\,[\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}]\,\mathcal{Y}\Leftrightarrow\mathcal{X}\not\asymp^{\mathfrak{A}}\mathcal{A}\wedge\mathcal{Y}\not\asymp^{\mathfrak{B}}\mathcal{B}.$$

Proposition 59. $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ is really a pointfree funcoid and for every $\mathcal{X}\in\mathfrak{A}$

$$\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle\mathcal{X}=\begin{cases}\mathcal{B} & \text{if }\mathcal{X}\not\asymp^{\mathfrak{A}}\mathcal{A};\\ 0^{\mathfrak{B}} & \text{if }\mathcal{X}\asymp^{\mathfrak{A}}\mathcal{A}.\end{cases}$$

Proof. Obvious. □

Proposition 60. Let $\mathfrak{A}$ and $\mathfrak{B}$ are bounded posets, $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$, $\mathcal{A}\in\mathfrak{A}$, $\mathcal{B}\in\mathfrak{B}$. Then

$$f\subseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\Leftrightarrow\operatorname{dom}f\subseteq\mathcal{A}\wedge\operatorname{im}f\subseteq\mathcal{B}.$$

Proof. If $f\subseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ then $\operatorname{dom}f\subseteq\operatorname{dom}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})\subseteq\mathcal{A}$, $\operatorname{im}f\subseteq\operatorname{im}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})\subseteq\mathcal{B}$. If $\operatorname{dom}f\subseteq\mathcal{A}\wedge\operatorname{im}f\subseteq\mathcal{B}$ then

$$\forall\mathcal{X}\in\mathfrak{A},\mathcal{Y}\in\mathfrak{B}: (\mathcal{X}\,[f]\,\mathcal{Y}\Rightarrow\mathcal{X}\not\asymp^{\mathfrak{A}}\mathcal{A}\wedge\mathcal{Y}\not\asymp^{\mathfrak{B}}\mathcal{B});$$

consequently $f\subseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$. □

Theorem 61. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. For every $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $\mathcal{A}\in\mathfrak{A}$, $\mathcal{B}\in\mathfrak{B}$

$$f\cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})=I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})}\circ f\circ I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}.$$

Proof. From above $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is a (complete) lattice.

$h \overset{\text{def}}{=} I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})} \circ f \circ I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}$. For every $\mathcal{X} \in \mathfrak{A}$

$$\langle h \rangle \mathcal{X} = \left\langle I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})} \right\rangle \langle f \rangle \left\langle I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} \right\rangle \mathcal{X} = \mathcal{B} \cap^{\mathfrak{B}} \langle f \rangle (\mathcal{A} \cap^{\mathfrak{A}} \mathcal{X}).$$

From this, as easy to show, $h \subseteq f$ and $h \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. If $g \subseteq f \wedge g \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ for a $g \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ then $\operatorname{dom} g \subseteq \mathcal{A}$, $\operatorname{im} g \subseteq \mathcal{B}$,

$$\langle g \rangle \mathcal{X} = \mathcal{B} \cap^{\mathfrak{B}} \langle g \rangle (\mathcal{A} \cap^{\mathfrak{A}} \mathcal{X}) \subseteq \mathcal{B} \cap^{\mathfrak{B}} \langle f \rangle (\mathcal{A} \cap^{\mathfrak{A}} \mathcal{X}) = \left\langle I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})} \right\rangle \langle f \rangle \left\langle I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} \right\rangle \mathcal{X} = \langle h \rangle \mathcal{X},$$

$g \subseteq h$. So $h = f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$. □

Corollary 62. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. For every $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}$ we have $f|_{\mathcal{A}} = f \cap (\mathcal{A} \times^{\mathsf{FCD}} 1^{\mathfrak{B}})$.

Proof. $f \cap (\mathcal{A} \times^{\mathsf{FCD}} 1^{\mathfrak{B}}) = I_{1^{\mathfrak{B}}}^{\mathsf{FCD}(\mathfrak{B})} \circ f \circ I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} = f \circ I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} = f|_{\mathcal{A}}$. □

Corollary 63. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. For every $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$ we have

$$f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \neq 0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \Leftrightarrow \mathcal{A} \, [f] \, \mathcal{B}.$$

Proof. $f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \neq 0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \Leftrightarrow \langle f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (\mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}) \rangle 1^{\mathfrak{A}} \neq 0^{\mathfrak{B}} \Leftrightarrow \left\langle I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})} \circ f \circ I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} \right\rangle 1^{\mathfrak{A}} \neq 0^{\mathfrak{B}} \Leftrightarrow \left\langle I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})} \right\rangle \langle f \rangle \left\langle I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} \right\rangle 1^{\mathfrak{A}} \neq 0^{\mathfrak{B}} \Leftrightarrow \mathcal{B} \cap^{\mathfrak{B}} \langle f \rangle (\mathcal{A} \cap^{\mathfrak{A}} 1^{\mathfrak{A}}) \neq 0^{\mathfrak{B}} \Leftrightarrow \mathcal{B} \cap^{\mathfrak{B}} \langle f \rangle \mathcal{A} \neq 0^{\mathfrak{B}} \Leftrightarrow \mathcal{A} \, [f] \, \mathcal{B}$. □

Theorem 64. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. Then the poset $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is separable.

Proof. Let $f, g \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $f \neq g$. By the theorem 12 $[f] \neq [g]$. That is there exist $x, y \in \mathfrak{A}$ such that $x \, [f] \, y \not\Leftrightarrow x \, [g] \, y$ that is $f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (x \times^{\mathsf{FCD}} y) \neq 0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \not\Leftrightarrow g \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (x \times^{\mathsf{FCD}} y) \neq 0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}$. Thus $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is separable. □

Theorem 65. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices. If $S \in \mathscr{P}(\mathfrak{A} \times \mathfrak{B})$ then

$$\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} = \bigcap^{\mathfrak{A}} \operatorname{dom} S \times^{\mathsf{FCD}} \bigcap^{\mathfrak{B}} \operatorname{im} S.$$

Proof. If $x \in \operatorname{atoms}^{\mathfrak{A}}$ then by the theorem 56

$$\left\langle \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} \right\rangle x = \bigcap^{\mathfrak{B}} \{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \}.$$

If $x \cap^{\mathfrak{A}} \bigcap^{\mathfrak{A}} \operatorname{dom} S \neq 0^{\mathfrak{A}}$ then

$$\forall (\mathcal{A}; \mathcal{B}) \in S: (x \cap^{\mathfrak{A}} \mathcal{A} \neq 0^{\mathfrak{A}} \wedge \langle \mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B} \rangle x = \mathcal{B});$$
$$\{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \} = \operatorname{im} S;$$

if $x \cap^{\mathfrak{A}} \bigcap^{\mathfrak{A}} \operatorname{dom} S = 0^{\mathfrak{A}}$ then

$$\exists (\mathcal{A}; \mathcal{B}) \in S: (x \cap^{\mathfrak{A}} \mathcal{A} = 0^{\mathfrak{A}} \wedge \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = 0^{\mathfrak{B}});$$
$$\{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \} \ni 0^{\mathfrak{B}}.$$

So

$$\left\langle \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} \right\rangle x = \begin{cases} \bigcap^{\mathfrak{B}} \operatorname{im} S & \text{if } x \cap^{\mathfrak{A}} \bigcap^{\mathfrak{A}} \operatorname{dom} S \neq 0^{\mathfrak{A}}; \\ 0^{\mathfrak{B}} & \text{if } x \cap^{\mathfrak{A}} \bigcap^{\mathfrak{A}} \operatorname{dom} S = 0^{\mathfrak{A}}. \end{cases}$$

From this by corollary 53 (taking in account 47 in [3]) follows the statement of the theorem. □

Corollary 66. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices.

For every $\mathcal{A}_0, \mathcal{A}_1 \in \mathfrak{A}$ and $\mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{B}$

$$(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = (\mathcal{A}_0 \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}_1).$$

Proof. $(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \{\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0, \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1\}$ what is by the last theorem equal to $(\mathcal{A}_0 \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}_1)$. □

Theorem 67. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices. If $\mathcal{A} \in \mathfrak{A}$ then $\mathcal{A} \times^{\mathsf{FCD}}$ is a complete homomorphism of the lattice $\mathfrak{A}$ to a the lattice $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$, if also $\mathcal{A} \neq 0^{\mathfrak{A}}$ then it is an order embedding.

Proof. Let $S \in \mathscr{P}\mathfrak{A}$, $X \in \mathfrak{Z}_0$, $x \in \operatorname{atoms}^{\mathfrak{A}}$.

$$\begin{aligned} \left\langle \bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle S \right\rangle X &= \bigcup^{\mathfrak{B}} \{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle X \mid \mathcal{B} \in S\} \\ &= \begin{cases} \bigcup^{\mathfrak{B}} S & \text{if } X \cap^{\mathfrak{A}} \mathcal{A} \neq 0^{\mathfrak{A}}; \\ 0^{\mathfrak{B}} & \text{if } X \cap^{\mathfrak{A}} \mathcal{A} = 0^{\mathfrak{A}} \end{cases} \\ &= \left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigcup^{\mathfrak{B}} S \right\rangle X. \end{aligned}$$

Thus $\bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle S = \mathcal{A} \times^{\mathsf{FCD}} \bigcup^{\mathfrak{B}} S$ by the theorem 25 (taking in account obvious 20 in [3]).

$$\begin{aligned} \left\langle \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle S \right\rangle x &= \bigcap^{\mathfrak{B}} \{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid \mathcal{B} \in S\} \\ &= \begin{cases} \bigcap^{\mathfrak{B}} S & \text{if } x \cap^{\mathfrak{A}} \mathcal{A} \neq 0^{\mathfrak{A}}; \\ 0^{\mathfrak{B}} & \text{if } x \cap^{\mathfrak{A}} \mathcal{A} = 0^{\mathfrak{A}} \end{cases} \\ &= \left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigcap^{\mathfrak{B}} S \right\rangle x. \end{aligned}$$

Thus $\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle S = \mathcal{A} \times^{\mathsf{FCD}} \bigcap^{\mathfrak{B}} S$ by the theorem 56.

If $\mathcal{A} \neq 0^{\mathfrak{A}}$ then obviously the function $\mathcal{A} \times^{\mathsf{FCD}}$ is injective. □

Proposition 68. Let $\mathfrak{A}$ is a meet-semilattice and $\mathfrak{B}$ is a poset with least element. If a is an atom of $\mathfrak{A}$, $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ then $f|_a = a \times^{\mathsf{FCD}} \langle f \rangle a$.

Proof. Let $\mathcal{X} \in \mathfrak{A}$.

$$\mathcal{X} \cap^{\mathfrak{A}} a \neq 0^{\mathfrak{A}} \Rightarrow \langle f|_a \rangle \mathcal{X} = \langle f \rangle a, \quad \mathcal{X} \cap^{\mathfrak{A}} a = 0^{\mathfrak{A}} \Rightarrow \langle f|_a \rangle \mathcal{X} = 0^{\mathfrak{B}}.$$ □

3.10 Atomic pointfree funcoids

Theorem 69. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. A $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is an atom of the poset $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ iff there exist $a \in \operatorname{atoms}^{\mathfrak{A}}$ and $b \in \operatorname{atoms}^{\mathfrak{B}}$ such that $f = a \times^{\mathsf{FCD}} b$.

Proof. $\mathfrak{A}$ and $\mathfrak{B}$ are atomic by the theorem 47 in [3].

⇒. Let f is an atom of the poset $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$. Let's get elements $a \in \operatorname{atoms}^{\mathfrak{A}} \operatorname{dom} f$ and $b \in \operatorname{atoms}^{\mathfrak{B}} \langle f \rangle a$. Then for every $\mathcal{X} \in \mathfrak{A}$

$$\mathcal{X} \asymp^{\mathfrak{A}} a \Rightarrow \langle a \times^{\mathsf{FCD}} b \rangle \mathcal{X} = 0^{\mathfrak{B}} \subseteq \langle f \rangle \mathcal{X}, \quad \mathcal{X} \not\asymp^{\mathfrak{A}} a \Rightarrow \langle a \times^{\mathsf{FCD}} b \rangle \mathcal{X} = b \subseteq \langle f \rangle \mathcal{X}.$$

So $a \times^{\mathsf{FCD}} b \subseteq f$; because f is atomic we have $f = a \times^{\mathsf{FCD}} b$.

⇐. Let $a \in \operatorname{atoms}^{\mathfrak{A}}$, $b \in \operatorname{atoms}^{\mathfrak{B}}$, $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$. If $b \asymp^{\mathfrak{B}} \langle f \rangle a$ then $\neg(a\ [f]\ b)$, $f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (a \times^{\mathsf{FCD}} b) = 0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}$ (because $\mathfrak{A}$ and $\mathfrak{B}$ are bounded meet-semilattices); if $b \subseteq \langle f \rangle a$ then $\forall \mathcal{X} \in \mathfrak{A}: (\mathcal{X} \not\asymp^{\mathfrak{A}} a \Rightarrow \langle f \rangle \mathcal{X} \supseteq b)$, $f \supseteq a \times^{\mathsf{FCD}} b$. Consequently $f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (a \times^{\mathsf{FCD}} b) = 0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \vee f \supseteq a \times^{\mathsf{FCD}} b$; that is $a \times^{\mathsf{FCD}} b$ is an atomic filter object. □

Theorem 70. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. Then $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is atomic.

Proof. Let $f \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $f \neq 0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}$. Then $\operatorname{dom} f \neq 0^{\mathfrak{A}}$, thus exists $a \in \operatorname{atoms}^{\mathfrak{A}} \operatorname{dom} f$. So $\langle f \rangle a \neq 0^{\mathfrak{B}}$ thus exists $b \in \operatorname{atoms}^{\mathfrak{B}} \langle f \rangle a$. Finally the atomic pointfree funcoid $a \times^{\mathsf{FCD}} b \subseteq f$. □

Theorem 71. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. Then the poset $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is separable.

Proof. Let $f, g \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$, $f \subset g$. Then taking in account the theorem 51 exists $a \in \operatorname{atoms}^{\mathfrak{A}}$ such that $\langle f \rangle a \subset \langle g \rangle a$. By corollary 17 in [3] $\mathfrak{B}$ is atomically separable. So exists $b \in \operatorname{atoms}^{\mathfrak{B}}$ such that $\langle f \rangle a \cap^{\mathfrak{B}} b = 0^{\mathfrak{B}}$ and $b \subseteq \langle g \rangle a$. For every $x \in \operatorname{atoms}^{\mathfrak{A}}$

$$\langle f \rangle a \cap^{\mathfrak{B}} \langle a \times^{\mathsf{FCD}} b \rangle a = \langle f \rangle a \cap^{\mathfrak{B}} b = 0^{\mathfrak{B}},$$
$$x \neq a \Rightarrow \langle f \rangle x \cap^{\mathfrak{B}} \langle a \times^{\mathsf{FCD}} b \rangle x = \langle f \rangle x \cap^{\mathfrak{B}} 0^{\mathfrak{B}} = 0^{\mathfrak{B}}.$$

Thus $\langle f \rangle x \cap^{\mathfrak{B}} \langle a \times b \rangle x = 0^{\mathfrak{B}}$ and consequently $f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (a \times^{\mathsf{FCD}} b) = 0^{\mathfrak{B}}$.

$$\langle a \times^{\mathsf{FCD}} b \rangle a = b \subseteq \langle g \rangle a,$$
$$x \neq a \Rightarrow \langle a \times^{\mathsf{FCD}} b \rangle x = 0^{\mathfrak{B}} \subseteq \langle g \rangle a.$$

Thus $\langle a \times^{\mathsf{FCD}} b \rangle x \subseteq \langle g \rangle x$ and consequently $a \times^{\mathsf{FCD}} b \subseteq g$.

So the lattice of funcoids is separable by the theorem 19 in [3]. □

Corollary 72. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. The poset $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is:

1. separable;
2. atomically separable;
3. conforming to Wallman's disjunction property.

Proof. By the theorem 22 in [3]. □

Remark 73. For more ways to characterize (atomic) separability of the lattice of pointfree funcoids see [3], subsections "Separation subsets and full stars" and "Atomically separable lattices".

Corollary 74. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices. The poset $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is an atomistic lattice.

Proof. By the theorem 30 $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is a complete lattice. Let $f \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$. Suppose contrary to the statement to be proved that $\bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} f \subset f$. Then exists $a \in \operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} f$ such that $a \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} f = 0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}$ what is impossible. □

Proposition 75. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices.

$\operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g) = \operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} f \cup \operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g$ for every $f, g \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$.

Proof. $(a \times^{\mathsf{FCD}} b) \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g) \neq \emptyset \Leftrightarrow a \left[f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g \right] b \Leftrightarrow a [f] b \vee a [g] b \Leftrightarrow (a \times^{\mathsf{FCD}} b) \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} f \neq 0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \vee (a \times^{\mathsf{FCD}} b) \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g \neq 0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}$ for every $a \in \operatorname{atoms}^{\mathfrak{A}}$ and $b \in \operatorname{atoms}^{\mathfrak{B}}$ (used the corollary 63 and theorem 32). □

Theorem 76. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices. For every $f, g, h \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$, $R \in \mathscr{P}\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$:

1. $f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (g \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} h) = (f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} g) \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} h)$;
2. $f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R = \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \langle f \cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \rangle R$.

Proof. We will take in account that the lattice of funcoids is an atomistic lattice (corollary 74). To be concise I will write atoms instead of $\operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}$ and $\cap$ and $\cup$ instead of $\cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}$ and $\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}$.

1. $\operatorname{atoms}(f \cap (g \cup h)) = \operatorname{atoms} f \cap \operatorname{atoms}(g \cup h) = \operatorname{atoms} f \cap (\operatorname{atoms} g \cup \operatorname{atoms} h) = (\operatorname{atoms} f \cap \operatorname{atoms} g) \cup (\operatorname{atoms} f \cap \operatorname{atoms} h) = \operatorname{atoms}(f \cap g) \cup \operatorname{atoms}(f \cap h) = \operatorname{atoms}((f \cap g) \cup (f \cap h))$.

2. $\operatorname{atoms}\left(f \cup \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R\right) = \operatorname{atoms} f \cup \operatorname{atoms} \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R = \operatorname{atoms} f \cup \bigcap \langle \operatorname{atoms} \rangle R = \bigcap \langle (\operatorname{atoms} f) \cup \rangle \langle \operatorname{atoms} \rangle R = \bigcap \langle \operatorname{atoms} \rangle \langle f \cup \rangle R = \operatorname{atoms} \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \langle f \cup \rangle R$. (Used the following equality.)

$$\begin{aligned}
\langle (\operatorname{atoms} f) \cup \rangle \langle \operatorname{atoms} \rangle R &= \\
\{ (\operatorname{atoms} f) \cup A \mid A \in \langle \operatorname{atoms} \rangle R \} &= \\
\{ (\operatorname{atoms} f) \cup A \mid \exists C \in R : A = \operatorname{atoms} C \} &= \\
\{ (\operatorname{atoms} f) \cup (\operatorname{atoms} C) \mid C \in R \} &= \\
\{ \operatorname{atoms}(f \cup C) \mid C \in R \} &= \\
\{ \operatorname{atoms} B \mid \exists C \in R : B = f \cup C \} &= \\
\{ \operatorname{atoms} B \mid B \in \langle f \cup \rangle R \} &= \\
\langle \operatorname{atoms} \rangle \langle f \cup \rangle. &
\end{aligned}$$

□

Corollary 77. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices. Then $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is a co-brouwerian lattice.

Proposition 78. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are sets of filters over some boolean lattices and $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$, $g \in \mathsf{FCD}(\mathfrak{B}; \mathfrak{C})$. Let $\mathfrak{B}$ is an atomic poset. Then

$\operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{C})}(g \circ f) = \left\{ x \times^{\mathsf{FCD}} z \mid x \in \operatorname{atoms}^{\mathfrak{A}}, z \in \operatorname{atoms}^{\mathfrak{C}}, \exists y \in \operatorname{atoms}^{\mathfrak{B}} : (x \times^{\mathsf{FCD}} y \in \operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} f \wedge y \times^{\mathsf{FCD}} z \in \operatorname{atoms}^{\mathsf{FCD}(\mathfrak{B};\mathfrak{C})} g) \right\}$.

Proof. $(x \times^{\mathsf{FCD}} z) \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{C})} (g \circ f) \neq 0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{C})} \Leftrightarrow x \, [g \circ f] \, z \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{B}} : (x \, [f] \, y \wedge y \, [g] \, z) \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{B}} : ((x \times^{\mathsf{FCD}} y) \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} f \neq 0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \wedge (y \times^{\mathsf{FCD}} z) \cap^{\mathsf{FCD}(\mathfrak{B};\mathfrak{C})} g \neq 0^{\mathsf{FCD}(\mathfrak{B};\mathfrak{C})})$ (were used the corollary 63 and theorem 36). □

3.11 Complete pointfree funcoids

Definition 79. Let $\mathfrak{A}$ and $\mathfrak{B}$ are posets. A pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is *complete*, when for every $S \in \mathscr{P}\mathfrak{A}$ whenever both $\bigcup^{\mathfrak{A}} S$ and $\bigcup^{\mathfrak{B}} \langle \langle f \rangle \rangle S$ are defined we have

$$\langle f \rangle \bigcup^{\mathfrak{A}} S = \bigcup^{\mathfrak{B}} \langle \langle f \rangle \rangle S.$$

Proposition 80. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. A pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is complete iff $\langle f \rangle a = \bigcup^{\mathfrak{B}} \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{A}} a$ for every $a \in \mathfrak{A}$.

Proof. Direct implication is obvious. The reverse implication follows from that $\mathfrak{A}$ is atomistic. □

Remark 81. Let $\mathfrak{Z}_0$ and $\mathfrak{Z}_1$ are join-semilattices with least elements. I will call *pointfree generalized closure* such a function $\alpha \in (\mathfrak{Z}_1)^{\mathfrak{Z}_0}$ that

1. $\alpha 0^{\mathfrak{Z}_0} = 0^{\mathfrak{Z}_1}$;
2. $\forall I, J \in \mathfrak{Z} : \alpha(I \cup^{\mathfrak{Z}_0} J) = \alpha I \cup^{\mathfrak{Z}_1} \alpha J$.

Definition 82. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices. I will call a *co-complete pointfree funcoid* a pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ such that $\langle f \rangle|_{\mathfrak{Z}_0}$ is a pointfree generalized closure.

Proposition 83. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices. Co-complete pointfree funcoids $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ bijectively correspond to pointfree generalized closures $\mathfrak{Z}_1^{\mathfrak{Z}_0}$, where the bijection is $f \mapsto \langle f \rangle|_{\mathfrak{Z}_0}$.

Proof. It follows from the theorem 26. □

Theorem 84. Let $(\mathfrak{A};\mathfrak{Z}_0)$ is semifiltered, star-separable, down-aligned filtrator with finitely meet closed, join-closed, and separable core, where $\mathfrak{Z}_0$ is a complete boolean lattice and both $\mathfrak{Z}_0$ and $\mathfrak{A}$ are atomistic lattices.

Let $(\mathfrak{B};\mathfrak{Z}_1)$ is a star-separable filtrator.

The following conditions are equivalent for every pointfree funcoid $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$:

1. f^{-1} is co-complete;
2. $\forall S\in\mathscr{P}\mathfrak{A}, J\in\mathfrak{Z}_1\colon (\bigcup^{\mathfrak{A}}S\,[f]\,J\Rightarrow\exists\mathcal{I}\in S\colon\mathcal{I}\,[f]\,J)$;
3. $\forall S\in\mathscr{P}\mathfrak{Z}_0, J\in\mathfrak{Z}_1\colon (\bigcup^{\mathfrak{Z}_0}S\,[f]\,J\Rightarrow\exists I\in S\colon I\,[f]\,J)$;
4. f is complete;
5. $\forall S\in\mathscr{P}\mathfrak{Z}_0\colon \langle f\rangle\bigcup^{\mathfrak{Z}_0}S=\bigcup^{\mathfrak{B}}\langle\langle f\rangle\rangle S$.

Proof. First note that the theorem 53 in [3] applies to the filtrator $(\mathfrak{A};\mathfrak{Z}_0)$.

(3)⇒(1). For every $S\in\mathscr{P}\mathfrak{Z}_0$, $J\in\mathfrak{Z}_1$

$$\bigcup^{\mathfrak{Z}_0}S\cap^{\mathfrak{A}}\langle f^{-1}\rangle J\neq 0^{\mathfrak{A}}\Rightarrow\exists I\in S\colon I\cap^{\mathfrak{A}}\langle f^{-1}\rangle J\neq 0^{\mathfrak{A}},\tag{9}$$

consequently by the theorem 53 in [3] we have $\langle f^{-1}\rangle J\in\mathfrak{Z}_0$.

(1)⇒(2). For every $S\in\mathscr{P}\mathfrak{A}$, $J\in\mathfrak{Z}_1$ we have $\langle f^{-1}\rangle J\in\mathfrak{Z}_0$, consequently the formula (9) is true. From this follows (2).

(2)⇒(4). Let $\langle f\rangle\bigcup^{\mathfrak{Z}_0}S$ and $\bigcup^{\mathfrak{B}}\langle\langle f\rangle\rangle S$ are defined. $J\cap^{\mathfrak{B}}\langle f\rangle\bigcup^{\mathfrak{A}}S\neq 0^{\mathfrak{B}}\Leftrightarrow\bigcup^{\mathfrak{A}}S\,[f]\,J\Leftrightarrow\exists\mathcal{I}\in S\colon \mathcal{I}\,[f]\,J\Leftrightarrow\exists\mathcal{I}\in S\colon J\cap^{\mathfrak{B}}\langle f\rangle\mathcal{I}\neq 0^{\mathfrak{B}}\Leftrightarrow J\cap^{\mathfrak{B}}\bigcup^{\mathfrak{B}}\langle\langle f\rangle\rangle S\neq 0^{\mathfrak{B}}$ (used the theorem 53 in [3]). Thus $\langle f\rangle\bigcup^{\mathfrak{A}}S=\bigcup^{\mathfrak{B}}\langle\langle f\rangle\rangle S$ by star-separability of $(\mathfrak{B};\mathfrak{Z}_1)$.

(5)⇒(3). Let $\langle f\rangle\bigcup^{\mathfrak{Z}_0}S$ is defined. Then $\bigcup^{\mathfrak{B}}\langle\langle f\rangle\rangle S$ is also defined because $\langle f\rangle\bigcup^{\mathfrak{Z}_0}S=\bigcup^{\mathfrak{B}}\langle\langle f\rangle\rangle S$. Then $\bigcup^{\mathfrak{Z}_0}S\,[f]\,J\Leftrightarrow J\cap^{\mathfrak{B}}\langle f\rangle\bigcup^{\mathfrak{Z}_0}S\neq 0^{\mathfrak{B}}\Leftrightarrow J\cap^{\mathfrak{B}}\bigcup^{\mathfrak{B}}\langle\langle f\rangle\rangle S\neq 0^{\mathfrak{B}}$ what by the theorem 53 in [3] equivalent to $\exists I\in S\colon J\cap^{\mathfrak{B}}\langle f\rangle I\neq 0^{\mathfrak{B}}$ that is $\exists I\in S\colon I\,[f]\,J$.

(2)⇒(3), (4)⇒(5). By join-closedness of the core of $(\mathfrak{A};\mathfrak{Z}_0)$. □

Theorem 85. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices and $\mathfrak{Z}_0$ is a complete boolean lattice. If R is a set of co-complete pointfree funcoids in $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ then $\bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R$ is a co-complete pointfree funcoid.

Proof. First, conditions of the theorem 84 apply.

Let R is a set of co-complete pointfree funcoids. Then for every $X\in\mathfrak{Z}_0$

$$\langle\bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R\rangle X=\bigcup^{\mathfrak{Z}_1}\{\langle f\rangle X\mid f\in R\}\in\mathfrak{Z}_1$$

(used the theorem 30). □

Let $\mathfrak{A}$ and $\mathfrak{B}$ are posets with least elements. I will denote $\mathrm{Compl}\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $\mathrm{CoCompl}\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ the sets of complete and co-complete funcoids correspondingly from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ with least elements.

Proposition 86.

1. Let $f\in\mathrm{Compl}\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $g\in\mathrm{Compl}\mathsf{FCD}(\mathfrak{B};\mathfrak{C})$ where $\mathfrak{A}$ and $\mathfrak{C}$ are posets with least elements and $\mathfrak{B}$ is a complete lattice. Then $g\circ f\in\mathrm{Compl}\mathsf{FCD}(\mathfrak{A};\mathfrak{C})$.
2. Let $f\in\mathrm{CoCompl}\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $g\in\mathrm{CoCompl}\mathsf{FCD}(\mathfrak{B};\mathfrak{C})$ where $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ are posets with least elements and $(\mathfrak{A};\mathfrak{Z}_0)$, $(\mathfrak{B};\mathfrak{Z}_1)$, $(\mathfrak{C};\mathfrak{Z}_2)$ are filtrators. Then $g\circ f\in\mathrm{CoCompl}\mathsf{FCD}(\mathfrak{A};\mathfrak{C})$.

Proof.

1. Let $\bigcup^{\mathfrak{A}}S$ and $\bigcup^{\mathfrak{C}}\langle\langle g\circ f\rangle\rangle S$ are defined. Then

$$\langle g\circ f\rangle\bigcup^{\mathfrak{A}}S=\langle g\rangle\langle f\rangle\bigcup^{\mathfrak{A}}S=\langle g\rangle\bigcup^{\mathfrak{B}}\langle\langle f\rangle\rangle S=\bigcup^{\mathfrak{C}}\langle\langle g\rangle\rangle\langle\langle f\rangle\rangle S=\bigcup^{\mathfrak{C}}\langle\langle g\circ f\rangle\rangle S.$$

2. $\langle g\circ f\rangle\mathfrak{Z}_0=\langle g\rangle\langle f\rangle\mathfrak{Z}_0\in\mathfrak{Z}_2$ because $\langle f\rangle\mathfrak{Z}_0\in\mathfrak{Z}_1$. □

Proposition 87. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices and $\mathfrak{Z}_0$ is a complete boolean lattice. Then $\mathrm{CoCompl}\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ (with induced order) is a complete lattice.

Proof. Follows from the theorem 85. □

3.12 Completion and co-completion

Definition 88. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices and $\mathfrak{Z}_1$ is a complete lattice.

Co-completion of a pointfree funcoid $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is pointfree funcoid $\mathrm{CoCompl}\, f$ defined by the formula (for every $X\in\mathfrak{Z}_0$)

$$\langle\mathrm{CoCompl}\, f\rangle X=\mathrm{Cor}\,\langle f\rangle X.$$

Proposition 89. Above defined co-completion always exists.

Proof. Existence of $\mathrm{Cor}\,\langle f\rangle X$ follows from completeness of $\mathfrak{Z}_1$.

We may apply the theorem 26 because

$$\mathrm{Cor}\,\langle f\rangle(X\cup^{\mathfrak{Z}_0}Y)=\mathrm{Cor}(\langle f\rangle X\cup^{\mathfrak{B}}\langle f\rangle Y)=\mathrm{Cor}\,\langle f\rangle X\cup^{\mathfrak{B}}\mathrm{Cor}\,\langle f\rangle Y$$

by the theorem 65 in [3]. □

Proposition 90. $\langle\mathrm{CoCompl}\, f\rangle X=\mathrm{Cor}'\,\langle f\rangle X$.

Proof. From the theorem 26 in [3]. (Existence of $\mathrm{Cor}'\,\langle f\rangle X$ follows from completeness of $\mathfrak{Z}_1$.) □

Obvious 91. Co-completion is always co-complete.

Obvious 92. For above defined always $\mathrm{CoCompl}\, f\subseteq f$.

3.13 Monovalued and injective pointfree funcoids

Definition 93. Let $\mathfrak{A}$ and $\mathfrak{B}$ are posets. Let $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$.

The pointfree funcoid f is:

- *monovalued* when $f\circ f^{-1}\subseteq 1^{\mathfrak{B}}$ if $\mathfrak{B}$ has a greatest element.
- *injective* when $f^{-1}\circ f\subseteq 1^{\mathfrak{A}}$ if $\mathfrak{A}$ has a greatest element.

Monovaluedness is dual of injectivity.

Proposition 94. Let $\mathfrak{A}$ and $\mathfrak{B}$ are posets. Let $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$.

The pointfree funcoid f is:

- monovalued iff $f\circ f^{-1}\subseteq I^{\mathsf{FCD}(\mathfrak{B})}_{\mathrm{im}\, f}$, if $\mathfrak{B}$ is a meet-semilattice;
- injective iff $f^{-1}\circ f\subseteq I^{\mathsf{FCD}(\mathfrak{A})}_{\mathrm{dom}\, f}$, if $\mathfrak{A}$ is a meet-semilattice.

Proof. It's enough to prove $f\circ f^{-1}\subseteq 1^{\mathfrak{B}}\Leftrightarrow f\circ f^{-1}\subseteq I^{\mathsf{FCD}(\mathfrak{B})}_{\mathrm{im}\, f}$.

⇐. Obvious.

⇒. Let $f\circ f^{-1}\subseteq 1^{\mathfrak{B}}$. Then $\langle f\circ f^{-1}\rangle x\subseteq x$ and $\langle f\circ f^{-1}\rangle x\subseteq\mathrm{im}\, f$. Thus $\langle f\circ f^{-1}\rangle x\subseteq x\cap^{\mathfrak{B}}\mathrm{im}\, f=\left\langle I^{\mathsf{FCD}(\mathfrak{B})}_{\mathrm{im}\, f}\right\rangle x$. Thus $f\circ f^{-1}\subseteq I^{\mathsf{FCD}(\mathfrak{B})}_{\mathrm{im}\, f}$. □

Obvious 95.

1. A morphism $(f;\mathfrak{A};\mathfrak{B};\mathcal{A};\mathcal{B})$ of the category of pointfree funcoid triples is monovalued iff the funcoid f is monovalued.

2. A morphism $(f; \mathfrak{A}; \mathfrak{B}; \mathcal{A}; \mathcal{B})$ of the category of pointfree funcoid triples is injective iff the funcoid f is injective.

Theorem 96. Let $\mathfrak{A}$ is an atomistic meet-semilattice, $\mathfrak{B}$ is a bounded meet-semilattice. The following statements are equivalent for every $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$:

1. f is monovalued.
2. $\forall a \in \operatorname{atoms}^{\mathfrak{A}}: \langle f \rangle a \in \operatorname{atoms}^{\mathfrak{B}} \cup \{0^{\mathfrak{B}}\}$.
3. $\forall i, j \in \mathfrak{A}: \langle f^{-1} \rangle (i \cap^{\mathfrak{B}} j) = \langle f^{-1} \rangle i \cap^{\mathfrak{A}} \langle f^{-1} \rangle j$.

Proof.

(2)⇒(3). Let $a \in \operatorname{atoms}^{\mathfrak{A}}$, $\langle f \rangle a = b$. Then because $b \in \operatorname{atoms}^{\mathfrak{B}} \cup \{0^{\mathfrak{B}}\}$

$$(i \cap^{\mathfrak{B}} j) \cap^{\mathfrak{B}} b \neq 0^{\mathfrak{B}} \Leftrightarrow i \cap^{\mathfrak{B}} b \neq 0^{\mathfrak{B}} \wedge j \cap^{\mathfrak{B}} b \neq 0^{\mathfrak{B}};$$
$$a\, [f]\, (i \cap^{\mathfrak{B}} j) \Leftrightarrow a\, [f]\, i \wedge a\, [f]\, j;$$
$$(i \cap^{\mathfrak{B}} j)\, [f^{-1}]\, a \Leftrightarrow i\, [f^{-1}]\, a \wedge j\, [f^{-1}]\, a;$$
$$a \cap^{\mathfrak{A}} \langle f^{-1} \rangle (i \cap^{\mathfrak{B}} j) \neq 0^{\mathfrak{A}} \Leftrightarrow a \cap^{\mathfrak{A}} \langle f^{-1} \rangle i \neq 0^{\mathfrak{A}} \wedge a \cap^{\mathfrak{A}} \langle f^{-1} \rangle j \neq 0^{\mathfrak{A}};$$
$$a \cap^{\mathfrak{A}} \langle f^{-1} \rangle (i \cap^{\mathfrak{B}} j) \neq 0^{\mathfrak{A}} \Leftrightarrow a \cap^{\mathfrak{A}} \langle f^{-1} \rangle i \cap^{\mathfrak{A}} \langle f^{-1} \rangle j \neq 0^{\mathfrak{A}};$$
$$\langle f^{-1} \rangle (i \cap^{\mathfrak{B}} j) = \langle f^{-1} \rangle i \cap^{\mathfrak{B}} \langle f^{-1} \rangle j.$$

(3)⇒(1). $\langle f^{-1} \rangle a \cap^{\mathfrak{A}} \langle f^{-1} \rangle b = \langle f^{-1} \rangle (a \cap^{\mathfrak{B}} b) = \langle f^{-1} \rangle 0^{\mathfrak{B}} = 0^{\mathfrak{A}}$ for every two distinct a, $b \in \operatorname{atoms}^{\mathfrak{B}}$. This is equivalent to $\neg(\langle f^{-1} \rangle a\, [f]\, b)$; $b \cap^{\mathfrak{B}} \langle f \rangle \langle f^{-1} \rangle a = 0$; $b \cap^{\mathfrak{B}} \langle f \circ f^{-1} \rangle a = 0^{\mathfrak{B}}$; $\neg(a\ [f \circ f^{-1}]\ b)$. So $a\ [f \circ f^{-1}]\ b \Rightarrow a = b$ for every a, $b \in \operatorname{atoms}^{\mathfrak{B}}$. This is possible only (corollary 53) when $f \circ f^{-1} \subseteq 1^{\mathfrak{B}}$.

¬(2)⇒¬(1). Suppose $\langle f \rangle a \notin \operatorname{atoms}^{\mathfrak{B}} \cup \{0^{\mathfrak{B}}\}$ for some $a \in \operatorname{atoms}^{\mathfrak{A}}$. Then there exist two atoms $p \neq q$ such that $\langle f \rangle a \supseteq p \wedge \langle f \rangle a \supseteq q$. Consequently $p \cap^{\mathfrak{B}} \langle f \rangle a \neq 0^{\mathfrak{B}}$; $a \cap^{\mathfrak{A}} \langle f^{-1} \rangle p \neq 0^{\mathfrak{A}}$; $a \subseteq \langle f^{-1} \rangle p$; $\langle f \circ f^{-1} \rangle p = \langle f \rangle \langle f^{-1} \rangle p \supseteq \langle f \rangle a \supseteq q$; $\langle f \circ f^{-1} \rangle p \nsubseteq p$ and $\langle f \circ f^{-1} \rangle p \neq 0^{\mathfrak{B}}$. So it cannot be $f \circ f^{-1} \subseteq 1^{\mathfrak{B}}$. □

Theorem 97. Let $(\mathfrak{B}; \mathfrak{Z}_1)$ is a primary filtrator over a meet-semilattice with greatest element and $(\mathfrak{A}; \mathfrak{Z}_0)$ is a primary filtrator over a boolean lattice. A pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is monovalued iff

$$\forall I, J \in \mathfrak{Z}_0: \langle f^{-1} \rangle (I \cap^{\mathfrak{Z}_1} J) = \langle f^{-1} \rangle I \cap^{\mathfrak{A}} \langle f^{-1} \rangle J.$$

Proof. $\mathfrak{A}$ and $\mathfrak{B}$ are complete lattices (corollary 8 in [3]).

$(\mathfrak{B}; \mathfrak{Z}_1)$ is a filtrator with separable core by the theorem 37 in [3].

$(\mathfrak{B}; \mathfrak{Z}_1)$ is finitely meet-closed by the theorem 29 in [3].

$\mathfrak{A}$ is an atomistic lattice by the theorem 48 in [3].

We are under conditions of the previous theorem.

⇒. Obvious.

⇐. $\langle f^{-1} \rangle (\mathcal{I} \cap^{\mathfrak{B}} \mathcal{J}) = \bigcap^{\mathfrak{A}} \langle \langle f^{-1} \rangle \rangle \operatorname{up}^{(\mathfrak{B}; \mathfrak{Z}_1)} (\mathcal{I} \cap^{\mathfrak{B}} \mathcal{J}) = \bigcap^{\mathfrak{A}} \langle \langle f^{-1} \rangle \rangle \{ I \cap^{\mathfrak{Z}_1} J \mid I \in \operatorname{up} \mathcal{I}, J \in \operatorname{up} \mathcal{J} \} = \bigcap^{\mathfrak{A}} \{ \langle f^{-1} \rangle (I \cap^{\mathfrak{Z}_1} J) \mid I \in \operatorname{up} \mathcal{I}, J \in \operatorname{up} \mathcal{J} \} = \bigcap^{\mathfrak{A}} \{ \langle f^{-1} \rangle I \cap^{\mathfrak{A}} \langle f^{-1} \rangle J \mid I \in \operatorname{up} \mathcal{I}, J \in \operatorname{up} \mathcal{J} \} = \bigcap^{\mathfrak{A}} \{ \langle f^{-1} \rangle I \mid I \in \operatorname{up} \mathcal{I} \} \cap^{\mathfrak{A}} \bigcap^{\mathfrak{A}} \{ \langle f^{-1} \rangle J \mid J \in \operatorname{up} \mathcal{J} \} = \langle f^{-1} \rangle \mathcal{I} \cap^{\mathfrak{A}} \langle f^{-1} \rangle \mathcal{J}$ (used theorem 25, theorem 34 in [3], theorem 15). □

3.14 Elements closed regarding a pointfree funcoid

Let $\mathfrak{A}$ is a poset with least element. Let $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{A})$.

Definition 98. Let's call *closed* regarding a pointfree funcoid f such element $a \in \mathfrak{A}$ that $\langle f \rangle a \subseteq a$.

Proposition 99. If i and j are closed (regarding a pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{A})$), S is a set of closed elements (regarding f), then

1. $i \cup^{\mathfrak{A}} j$ is a closed element, if $\mathfrak{A}$ is a separable starrish join-semilattice;

2. $\bigcap^{\mathfrak{A}} S$ is a closed element.

Proof. $\langle f\rangle(i\cup^{\mathfrak{A}} j)=\langle f\rangle i\cup^{\mathfrak{A}}\langle f\rangle j\subseteq i\cup^{\mathfrak{A}} j$ (theorem 15), $\langle f\rangle\bigcap^{\mathfrak{A}} S\subseteq\bigcap^{\mathfrak{A}}\langle\langle f\rangle\rangle S\subseteq\bigcap^{\mathfrak{A}} S$. Consequently the elements $i\cup^{\mathfrak{A}} j$ and $\bigcap^{\mathfrak{A}} S$ are closed. □

Proposition 100. If S is a set of elements closed regarding a complete pointfree funcoid f, then the element $\bigcup^{\operatorname{Src} f} S$ is also closed regarding our funcoid.

Proof. $\langle f\rangle\bigcup^{\operatorname{Src} f} S=\bigcup^{\operatorname{Dst} f}\langle\langle f\rangle\rangle S\subseteq\bigcup^{\operatorname{Dst} f} S$. □

3.15 Connectedness regarding a pointfree funcoid

Let $\mathfrak{A}$ is a poset with least element. Let $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{A})$.

Definition 101. An element $a\in\mathfrak{A}$ is called *connected* regarding a pointfree funcoid μ over $\mathfrak{A}$ when

$$\forall x,y\in\mathfrak{A}\setminus\{0^{\mathfrak{A}}\}:(x\cup^{\mathfrak{A}} y=a\Rightarrow x\,[\mu]\,y).$$

Proposition 102. Let $(\mathfrak{A};\mathfrak{Z})$ is a co-separable filtrator. An $A\in\mathfrak{Z}$ is connected regarding a funcoid μ iff

$$\forall X,Y\in\mathfrak{Z}\setminus\{0^{\mathfrak{Z}}\}:(X\cup^{\mathfrak{Z}} Y=A\Rightarrow X\,[\mu]\,Y).$$

Proof.

⇒. Obvious.

⇐. Follows from co-separability. □

Obvious 103. For $\mathfrak{A}$ being a set of filters over a boolean lattice, an element $a\in\mathfrak{A}$ is connected regarding a pointfree funcoid μ iff it is connected regarding the funcoid $\mu\cap^{\mathsf{FCD}}(a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{A})} a)$.

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